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On exact solutions for nonlinear Burger–Fisher equation using the Exp-function method

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A B S T R A C T

In this paper, we will apply the Exp-function method with the aid of Maple software to solve the nonlinear Burger-Fisher equation.

The Exp-function method is one of the most recent analytical methods which used to solve various types of nonlinear partial differential equations (PDEs). These nonlinear PDEs are transformed first into nonlinear ordinary differential equations (ODEs) and then by using the ansatz of the Exp-function method for balancing the highest order of linear and nonlinear terms in nonlinear ODE., we obtain the exact solution.

Keywords: Exact solutions; The Exp-function ansatz; nonlinear partial differential equations; nonlinear Burger-Fisher equation.

حول الموجية التامة لمعادلة كلاين–غوردن غير الخطية باستخدام طريقة الدالة الأسية

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الملخص

في هذه الورقة، سنطبق طريقة الدالة الأسية بمساعدة برنامج ميبل لحل معادلة Burger–Fisher غيرالخطية ،طريقة الدالة الأسية هي واحدة من أكثر الطرق التحليلية التي تستخدم لحل أنواع مختلفة من المعادلات الجزئية غير الخطية. هذه المعادلات التفاضلية الجزئية غير الخطية تحول أولاً إلى معادلات تفاضلية عادية غير خطية ومن ثم باستخدام فرضية طريقة الدالة الأسية لموازنة أعلى رتب لحدود خطية وغير خطية في المعادلة التفاضلية العادية غير الخطية، نتحصل على الحل التام.

الكلمات المفتاحية: الحلول التامة؛ فرضية الدالة الأسية؛ المعادلات التفاضلية الجزئية غير الخطية؛ معادلة Burger–Fisher غيرالخطية

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Introduction

In the recent years, searching for the exact solutions to the nonlinear partial differential equations is one of the most important subjects in plasma physics, mechanics, biology, chemistry, engineering, fluid mechanics optical fibers and etc. Study of exact solutions of nonlinear PDEs will help to study physical phenomenon [11], [16].

The problem of the study is that nonlinear Burger-Fisher equation is difficult to solve by direct methods. Thus, we resort to modern methods such as the Exp-function method to obtain exact solutions [8], [9].

The Exp-function method is one of the most recent analytical methods used for the solution of nonlinear PDEs [12]. This method has been used by many authors in a wide variety of physical problems and engineering applications to solve different types of nonlinear PDEs [9], [15].

The Exp-function method was first proposed by He and Wu in 2006 [10]. This method plays an important role in searching for analytic solutions of many nonlinear PDEs [2].

The importance of the study is the importance of PDEs in our practical life, as they describe physical phenomena in various sciences [11].

The Objective of the study is transform the nonlinear Burger-Fisher equation into nonlinear ODE with a new independent variable η and then applying the ansatz $u(\eta) = \frac{\sum_{n=-c}^{p} a_n \exp(n\eta)}{\sum_{m=-d}^{q} b_m \exp(m\eta)}$, where c, d, p and q are positive integers which could be freely chosen, a_n and b_m are unknown constants to be determined [3-7]. To determine the values of p and q we balance the highest order linear term in nonlinear ODE with the highest order nonlinear term, we obtain $p = q$. Similarly, to determine the values of c and d we balance the lowest order linear term in nonlinear ODE with the lowest order nonlinear term, we obtain $c = d$.

The relations $c = d$ and $p = q$ are the only that can be obtained through applying the Exp-function ansatz for all possible cases of nonlinear ODEs [8].

2. Description of the Exp-Function method

Assume that the general form of nonlinear PDE is

$$P(u, u_x, u_t, u_{xx}, u_{tt}, u_{xt}, \dots) = 0. \quad (2.1)$$

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Step 1. Using wave transformation

$$u(x,t) = u(\eta)\,;\ \eta = kx + \omega t\,, \tag{2.2}$$

where k and ω are constants, x is independent variable and t is parameter. Then Eq. (2.1) becomes nonlinear ODE as the following

$$Q(u, ku', \omega u', k^2u'', \omega^2u'', \ldots) = 0. \tag{2.3}$$

Step 2. Assume that the solution of Eq. (2.3) has the following form

$$u(\eta) = \frac{\sum_{n=-c}^{p} a_n \exp(n\eta)}{\sum_{m=-d}^{q} b_m \exp(m\eta)} = \frac{a_{-c} \exp(-c\eta) + \cdots + a_p \exp(p\eta)}{b_{-d} \exp(-d\eta) + \cdots + b_q \exp(q\eta)}\,, \tag{2.4}$$

where c, d, p and q are positive integers which are unknown could be freely chosen, a_n and b_m are unknown constants.

Step 3. Substituting Eq. (2.4) into Eq. (2.3).

Step 4. We determine the values of c and d by balancing the linear term of lowest order in Eq. (2.3) with the lowest order nonlinear term. Similarly, we determine the values of p and q by balancing the linear term of highest order in Eq. (2.3) with the highest order nonlinear term [4-7].

Step 5. We Collecting terms of the same term of each power of exp $(i\eta)$ where $(i = 0, \pm1, \pm2, \pm3, \ldots)$.

Step 6. We equating the coefficients of each power of exp $(i\eta)$ to zero.

Step 7. We solve a set of algebraic equations for determining the unknown constants.

Step 8. Substituting the solutions of Step 7, into Eq. (2.4) we have the exact solution.

3. Some general formulas

On using the ansatz given by Eq. (2.4), then the following derivatives are resulted

$$u'(\eta) = \frac{\tau_1 \exp[-(c+d)\eta] + \cdots + \sigma_1 \exp[(p+q)\eta]}{\varrho_1 \exp[(-2d)\eta] + \cdots + \Gamma_1 \exp[(2q)\eta]}\,, \tag{3.1}$$

$$u''(\eta) = \frac{\tau_2 \exp[-(c+3d)\eta] + \cdots + \sigma_2 \exp[(p+3q)\eta]}{\varrho_2 \exp[(-4d)\eta] + \cdots + \Gamma_2 \exp[(4q)\eta]}\,, \tag{3.2}$$

$$u'''(\eta) = \frac{\tau_3 \exp[-(c+7d)\eta] + \cdots + \sigma_3 \exp[(p+7q)\eta]}{\varrho_3 \exp[(-8d)\eta] + \cdots + \Gamma_3 \exp[(8q)\eta]}\,, \tag{3.3}$$

$$\vdots$$

$$u^{(r)}(\eta) = \frac{\tau_r \exp[-(c+(2^r-1)d)\eta] + \cdots + \sigma_r \exp[(p+(2^r-1)q)\eta]}{\varrho_r \exp[(-2^r d)\eta] + \cdots + \Gamma_r \exp[(2^r q)\eta]} \tag{3.4}$$

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where $\tau_r, \sigma_r, \varrho_r$ and Γ_r are all constants. And r-derivative of $u(\eta)$ is called the general formula for the derivative r-times [1].

4. The nonlinear Burger-Fisher equation

we consider the Burger–Fisher equation

$$u_t + uu_x + u_{xx} + u - u^2 = 0\,. \tag{4.1}$$

Using wave transformation in Eq. (2.2), therefore Eq. (4.1) becomes

$$\omega u' + kuu' + k^2u'' + u - u^2 = 0\,. \tag{4.2}$$

where the prime denotes the derivation with respect to η.

Using the ansatz (2.4) for balance procedure, the linear term of lowest order u'' in Eq. (4.2) with the lowest order nonlinear term uu'. we have

$$u' = \frac{-ca_{-c}(b_{-d})\exp[-(c+d)\eta] + \cdots}{(b_{-d})^2\exp[(-2d)\eta] + \cdots}\,, \tag{4.3}$$

$$u'' = \frac{ca_{-c}(b_{-d})^3(c+d)\exp[-(c+3d)\eta] + \cdots}{(b_{-d})^4\exp[(-4d)\eta] + \cdots}\,. \tag{4.4}$$

Let $c_1 = ca_{-c}(b_{-d})^3(c+d)$, $c_2 = (b_{-d})^4$.

Therefore Eq. (4.4), becomes

$$u'' = \frac{c_1\exp[-(c+3d)\eta] + \cdots}{c_2\exp[(-4d)\eta] + \cdots}\,, \tag{4.5}$$

$$uu' = \frac{-cb_{-d}(a_{-c})^2\exp[-(2c+d)\eta] + \cdots}{(b_{-d})^3\exp[(-3d)\eta] + \cdots}\,, \tag{4.6}$$

Let $c_3 = -cb_{-d}(a_{-c})^2$, $c_4 = (b_{-d})^3$.

Therefore Eq. (4.6) becomes

$$uu' = \frac{c_3\exp[-(2c+d)\eta] + \cdots}{c_4\exp[(-3d)\eta] + \cdots}\,. \tag{4.7}$$

Multiplying both numerator and denominator of the R.H.S of the Eq. (4.7) by $\exp[(-d\eta)]$, we get

$$uu' = \frac{c_3\exp[-(2c+2d)\eta] + \cdots}{c_4\exp[(-4d)\eta] + \cdots}\,. \tag{4.8}$$

where c_1, c_2, c_3 and c_4 are determined coefficients only for simplicity.

Balancing lowest order of Exp-function in Eqs.(4.5) and (4.8), we have

$$-(c+3d) = -(2c+2d)\,, \tag{4.9}$$

which leads to the result

$$2c - c = 3d - 2d \quad \Longrightarrow c = d. \tag{4.10}$$

Similarly, using the ansatz (2.4) for balance procedure, the linear term of highest order u'' in Eq. (4.2) with the highest order nonlinear term uu'. we have

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$$u' = \frac{\dots + pa_p b_q \exp[(p+q)\eta]}{\dots + (b_q)^2 \exp[(2q)\eta]} \ , \tag{4.11}$$

$$u'' = \frac{\dots + p(a_p)(b_q)^3(p+q)\exp[(p+3q)\eta]}{\dots + (b_q)^4 \exp[(4q)\eta]} \ , \tag{4.12}$$

Let $d_1 = p\big(a_p\big)(b_q)^3(p+q)$, $d_2 = (b_q)^4$.

Therefore Eq. (4.12) becomes

$$u'' = \frac{\dots + d_1 \exp[(p+3q)\eta]}{\dots + d_2 \exp[(4q)\eta]} \ , \tag{4.13}$$

$$uu' = \frac{\dots + p(a_p)^2 b_q \exp[(2p+q)\eta]}{\dots + (b_q)^3 \exp[(3q)\eta]} \ . \tag{4.14}$$

Let $d_3 = p(a_p)^2 b_q$, $d_4 = (b_q)^3$.

Therefore Eq. (4.14) becomes

$$uu' = \frac{\dots + d_3 \exp[(2p+q)\eta]}{\dots + d_4 \exp[(3q)\eta]} \ . \tag{4.15}$$

Multiplying both numerator and denominator of the R.H.S of the Eq. (4.15) by exp $[(q\eta)]$, we get

$$uu' = \frac{\dots + d_3 \exp[(2p+2q)\eta]}{\dots + d_4 \exp[(4q)\eta]} \ . \tag{4.16}$$

where d_1, d_2, d_3 and d_4 are determined coefficients only for simplicity. Balancing highest order of Exp-function in Eqs. (4.13) and (4.16), we have

$$p + 3q = 2p + 2q \ , \tag{4.17}$$

which leads to the result

$$2p - p = 3q - 2q \ \Longrightarrow \ p = q \ . \tag{4.18}$$

Now we study the following cases

Case 1.

Let $c = d = 1$ and $p = q = 1$, so Eq. (2.4) reduce to

$$u(\eta) = \frac{\sum_{n=-1}^{1} a_n \exp(n\eta)}{\sum_{m=-1}^{1} b_m \exp(m\eta)} = \frac{a_{-1}\exp(-\eta) + a_0 + a_1 \exp(\eta)}{b_{-1}\exp(-\eta) + b_0 + b_1 \exp(\eta)} \ . \tag{4.19}$$

There are some free parameters in Eq. (4.19), for simplicity we set $b_1 = 1$, therefore Eq. (4.19) becomes as the following

$$u(\eta) = \frac{a_{-1}\exp(-\eta) + a_0 + a_1 \exp(\eta)}{b_{-1}\exp(-\eta) + b_0 + \exp(\eta)} \ . \tag{4.20}$$

Substituting Eq. (4.20) into Eq. (4.2), and by the help of Maple, we have

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$$\frac{1}{A}[E_3\exp(3\eta)+E_2\exp(2\eta)+E_1\exp(\eta)+E_0+E_{-1}\exp(-\eta)+E_{-2}\exp(-2\eta)+E_{-3}\exp(-3\eta)]=0\,. \quad (4.21)$$

where

$$A=(b_{-1}\exp(-\eta)+b_0+\exp(\eta))^3\,,$$

$$E_3=-a_1{}^2+a_1\,,$$

$$E_2=-k^2a_1b_0+ka_1{}^2b_0+k^2a_0-ka_0a_1+\omega a_1b_0-a_1{}^2b_0-\omega a_0-2a_0a_1+2a_1b_0+a_0\,,$$

$$E_1=k^2a_1b_0{}^2-k^2a_0b_0-4k^2a_1b_{-1}+ka_0a_1b_0+2ka_1{}^2b_{-1}+\omega a_1b_0{}^2+4k^2a_{-1}-2ka_{-1}a_1-ka_0{}^2-\omega a_0b_0+2\omega a_1b_{-1}-2a_0a_1b_0-a_1{}^2b_{-1}+a_1b_0{}^2-2\omega a_{-1}-2a_{-1}a_1-a_0{}^2+2a_0b_0+2a_1b_{-1}+a_{-1}\,,$$

$$E_0=3k^2a_1b_{-1}b_0+3k^2a_{-1}b_0-6k^2a_0b_{-1}+3ka_0a_1b_{-1}+3\omega a_1b_{-1}b_0-3ka_{-1}a_0-3\omega a_{-1}b_0-2a_{-1}a_1b_0-a_0{}^2b_0-2a_0a_1b_{-1}+a_0b_0{}^2+2a_1b_{-1}b_0-2a_{-1}a_0+2a_{-1}b_0+2a_0b_{-1}\,,$$

$$E_{-1}=k^2a_{-1}b_0{}^2-k^2a_0b_{-1}b_0+4k^2a_1b_{-1}{}^2-4k^2a_{-1}b_{-1}-ka_{-1}a_0b_0+2ka_{-1}a_1b_{-1}+ka_0{}^2b_{-1}-\omega a_{-1}b_0{}^2+\omega a_0b_{-1}b_0+2\omega a_1b_{-1}{}^2-2ka_{-1}{}^2-2\omega a_{-1}b_{-1}-2a_{-1}a_0b_0-2a_{-1}a_1b_{-1}+a_{-1}b_0{}^2-a_0{}^2b_{-1}+2a_0b_{-1}b_0+a_1b_{-1}{}^2-a_{-1}{}^2+2a_{-1}b_{-1}\,,$$

$$E_{-2}=-k^2a_{-1}b_{-1}b_0+k^2a_0b_{-1}{}^2-ka_{-1}{}^2b_0+ka_{-1}a_0b_{-1}-\omega a_{-1}b_{-1}b_0+\omega a_0b_{-1}{}^2-a_{-1}{}^2b_0-2a_{-1}a_0b_{-1}+2a_{-1}b_{-1}b_0+a_0b_{-1}{}^2\,,$$

$$E_{-3}=-a_{-1}{}^2b_{-1}+a_{-1}b_{-1}{}^2\,,$$

Equating the coefficients of $\exp(i\eta)$; $(i=0,\pm1,\pm2,\pm3)$ to be zero, we have

$$\begin{cases} E_0=0\,, & E_{-1}=0\,,\\ E_1=0\,, & E_{-2}=0\,,\\ E_2=0\,, & E_{-3}=0\,,\\ & E_3=0\,. \end{cases} \quad (4.22)$$

Thus, we obtain a system of algebraic equations for $a_1, a_0, a_{-1}, b_{-1}, b_0, \omega$ and k. Solving the system of Eq. (4.22), simultaneously, by the help of Maple, we have

$$\begin{cases} a_0=a_0\,, & b_{-1}=-a_0{}^2+a_0b_0\,, & k=\dfrac{1}{2}\,,\\ b_0=b_0, & a_{-1}=0\,, \quad a_1=1\,, & \omega=\dfrac{-5}{4}\,. \end{cases} \quad (4.23)$$

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where a_0, b_0 and b_{-1} are free parameters. Let $a_0 = -1$ and $b_0 = -\frac{1}{4}$, therefore $b_{-1} = -\frac{3}{4}$

Substituting these results into Eq. (4.20), we obtain the following solitary exact solution

$$u(x,t) = \frac{-1 + \exp\left(\frac{1}{2}x - \frac{5}{4}t\right)}{-\frac{3}{4}\exp\left[-\left(\frac{1}{2}x - \frac{5}{4}t\right)\right] - \frac{1}{4} + \exp\left(\frac{1}{2}x - \frac{5}{4}t\right)}. \quad (4.24)$$

which is the solitary wave solution of the nonlinear Burger-Fisher equation when $c = d = 1$ **and** $\boldsymbol{p = q = 1}$**.**

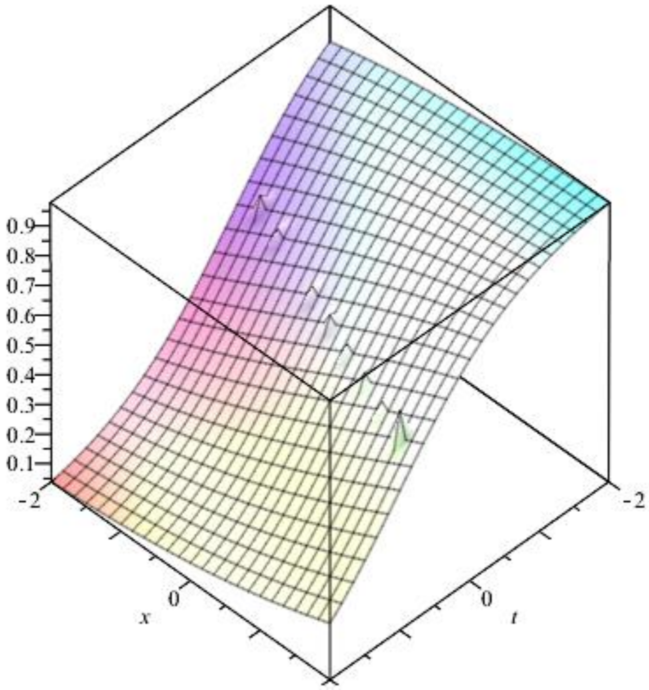

Fig 1: The solitary wave solution when $(b_{-1} = -\frac{3}{4}, k = \frac{1}{2}, \omega = \frac{-5}{4})$

Case 2.

Let $c = d = 1$ and $p = q = 2$, so Eq. (2.4) reduce to

$$u(\eta) = \frac{\sum_{n=-1}^{2} a_n \exp(n\eta)}{\sum_{m=-1}^{2} b_m \exp(m\eta)}$$

$$= \frac{a_{-1}\exp(-\eta) + a_0 + a_1\exp(\eta) + a_2\exp(2\eta)}{b_{-1}\exp(-\eta) + b_0 + b_1\exp(\eta) + b_2\exp(2\eta)}. \quad (4.25)$$

There are some free parameters in Eq. (4.25), for simplicity, we set $b_2 = 1$ and $b_1 = 0$, therefore Eq. (4.25) becomes as the following

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$$u(\eta)=\frac{a_{-1}\exp(-\eta)+a_0+a_1\exp(\eta)+a_2\exp(2\eta)}{b_{-1}\exp(-\eta)+b_0+\exp(2\eta)}\ . \qquad (4.26)$$

Substituting Eq. (4.26) into Eq. (4.2), and by the help of Maple, we have

$$\frac{1}{A}[E_6\exp(6\eta)+E_5\exp(5\eta)+E_4\exp(4\eta)+E_3\exp(3\eta)+E_2\exp(2\eta)$$
$$+E_1\exp(\eta)+E_0+E_{-1}\exp(-\eta)+E_{-2}\exp(-2\eta)\ +E_{-3}\exp(-3\eta)]$$
$$=0\,. \qquad (4.27)$$

where

$$A=(b_{-1}\exp(-\eta)+b_0+\exp(2\eta))^3\,,$$
$$E_6=a_2{}^2-a_2\,,$$
$$E_5=-k^2a_1+ka_1a_2+\omega a_1+2a_1a_2-a_1\,,$$
$$E_4=4k^2a_2b_0-2ka_2{}^2b_0-4k^2a_0+2ka_0a_2+ka_1{}^2-2\omega a_2b_0$$
$$+a_2{}^2b_0+2\omega a_0+2a_0a_2+a_1{}^2-2a_2b_0-a_0\,,$$
$$E_3=6k^2a_1b_0+9k^2a_2b_{-1}-3ka_1a_2b_0-3ka_2{}^2b_{-1}-9k^2a_{-1}$$
$$+3ka_{-1}a_2+3ka_0a_1-3\omega a_2b_{-1}+2a_1a_2b_0+a_2{}^2b_{-1}$$
$$+3\omega a_{-1}+2a_{-1}a_2+2a_0a_1-2a_1b_0-2a_2b_{-1}-a_{-1}\,,$$
$$E_2=-4k^2a_2b_0{}^2+4k^2a_0b_0+13k^2a_1b_{-1}-2ka_0a_2b_0-ka_1{}^2b_0$$
$$-5ka_1a_2b_{-1}-2\omega a_2b_0{}^2+4ka_{-1}a_1+2ka_0{}^2+2\omega a_0b_0$$
$$-\omega a_1b_{-1}+2a_0a_2b_0+a_1{}^2b_0+2a_1a_2b_{-1}-a_2b_0{}^2$$
$$+2a_{-1}a_1+a_0{}^2-2a_0b_0-2a_1b_{-1}\,,$$
$$E_1=-k^2a_1b_0{}^2-11k^2a_2b_{-1}b_0-2k^2a_{-1}b_0+13k^2a_0b_{-1}-ka_{-1}a_2b_0$$
$$-ka_0a_1b_0-4ka_0a_2b_{-1}-2ka_1{}^2b_{-1}-\omega a_1b_0{}^2$$
$$-5\omega a_2b_{-1}b_0+5ka_{-1}a_0+4\omega a_{-1}b_0+\omega a_0b_{-1}$$
$$+2a_{-1}a_2b_0+2a_0a_1b_0+2a_0a_2b_{-1}+a_1{}^2b_{-1}-a_1b_0{}^2$$
$$-2a_2b_{-1}b_0+2a_{-1}a_0-2a_{-1}b_0-2a_0b_{-1}\,,$$
$$E_0=-9k^2a_2b_{-1}{}^2+9k^2a_{-1}b_{-1}+3\omega a_{-1}b_{-1}+2a_0a_1b_{-1}+2a_{-1}a_1b_0$$
$$-3\omega a_2b_{-1}{}^2-2a_1b_{-1}b_0+2a_{-1}a_2b_{-1}-a_2b_{-1}{}^2$$
$$-3k^2a_1b_{-1}b_0-3\omega a_1b_{-1}b_0-3ka_{-1}a_2b_{-1}$$
$$-3ka_0a_1b_{-1}+a_{-1}{}^2-2a_{-1}b_{-1}+3ka_{-1}{}^2+a_0{}^2b_0$$
$$-a_0b_0{}^2\,,$$
$$E_{-1}=-k^2a_{-1}b_0{}^2+k^2a_0b_{-1}b_0-4k^2a_1b_{-1}{}^2+ka_{-1}a_0b_0$$
$$-2ka_{-1}a_1b_{-1}-ka_0{}^2b_{-2}+\omega a_{-1}b_0{}^2-\omega a_0b_{-1}b_0$$
$$-2\omega a_1b_{-1}{}^2+2a_{-1}a_0b_0+2a_{-1}a_1b_{-1}+a_0{}^2b_{-2}$$
$$-a_{-1}b_0{}^2-2a_0b_{-1}b_0-a_1b_{-1}{}^2\,,$$

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$$E_{-2} = k^2 a_{-1} b_{-1} b_0 - k^2 a_0 {b_{-1}}^2 + k {a_{-1}}^2 b_0 - k a_{-1} a_0 b_{-1} + \omega a_{-1} b_{-1} b_0 - \omega a_0 {b_{-1}}^2 + {a_{-1}}^2 b_0 + 2 a_{-1} a_0 b_{-1} - 2 a_{-1} b_{-1} b_0 - a_0 {b_{-1}}^2 ,$$

$$E_{-3} = {a_{-1}}^2 b_{-1} - a_{-1} {b_{-1}}^2 ,$$

Equating the coefficients of $\exp(i\eta)$; $(i = 6, 5, 4, \dots, -3)$ to zero, we have

$$\begin{cases} E_0 = 0 , & E_3 = 0 , & E_{-1} = 0 , \\ E_1 = 0 , & E_4 = 0 , & E_{-2} = 0 , \\ E_2 = 0 , & E_5 = 0 , & E_{-3} = 0 , \\ & E_6 = 0 . & \end{cases} \tag{4.28}$$

Thus, we obtain a system of algebraic equations for $a_2, a_1, a_0, a_{-1}, b_{-1}, b_0, \omega$ and k. Solving the system of Eq. (4.28), simultaneously, by the help of Maple, we get

$$\begin{cases} a_{-1} = b_{-1} , \quad a_1 = a_1 , \quad b_{-1} = b_{-1} , \quad k = -\dfrac{1}{2} , \\ a_0 = -{a_1}^2 , \quad a_2 = 0 , \quad b_0 = -\dfrac{{a_1}^3 - b_{-1}}{a_1} , \quad \omega = \dfrac{5}{4} . \end{cases} \tag{4.29}$$

where a_{-1}, b_{-1}, a_1, a_0 and b_0 are free parameters.

Let $a_{-1} = 3$, $a_1 = -2$ and $b_{-1} = 3$, therefore $b_0 = -\frac{11}{2}$ and $a_0 = -4$

Substituting these results into Eq. (4.26), we obtain the following solitary exact solution

$$u(x,t) = \frac{3 \exp\left(\frac{1}{2}x - \frac{5}{4}t\right) - 4 - 2 \exp\left(-\frac{1}{2}x + \frac{5}{4}t\right)}{3 \exp\left(\frac{1}{2}x - \frac{5}{4}t\right) - \frac{11}{2} + \exp\left(-x + \frac{5}{2}t\right)} . \tag{4.30}$$

which is the solitary wave solution of the nonlinear Burger-Fisher equation when $c = d = 1$ and $p = q = 2$.

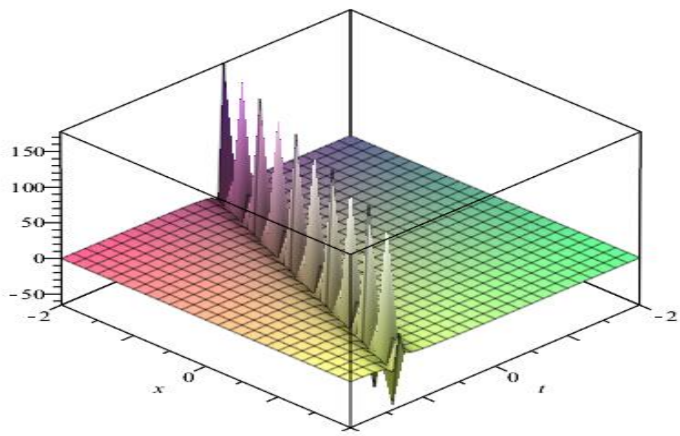

Fig 2: The solitary wave solution when

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$$\left(b_0 = -\frac{11}{2}, a_0 = -4, k = -\frac{1}{2}, \omega = \frac{5}{4}\right)$$

Case 3.

Let $c = d = 2$ and $p = q = 2$, so Eq. (2.4) reduce to

$$u(\eta) = \frac{\sum_{n=-2}^{2} a_n \exp(n\eta)}{\sum_{m=-2}^{2} b_m \exp(m\eta)}$$

$$= \frac{a_{-2}\exp(-2\eta) + a_{-1}\exp(-\eta) + a_0 + a_1\exp(\eta) + a_2\exp(2\eta)}{b_{-2}\exp(-2\eta) + b_{-1}\exp(-\eta) + b_0 + b_1\exp(\eta) + b_2\exp(2\eta)}. \quad (4.31)$$

There are some free parameters in Eq. (4.31), for simplicity, we set $b_{-1} = 0$, $b_1 = 0$ and $b_2 = 1$, therefore Eq. (4.31) becomes as the following

$$u(\eta) = \frac{a_{-2}\exp(-2\eta) + a_{-1}\exp(-\eta) + a_0 + a_1\exp(\eta) + a_2\exp(2\eta)}{b_{-2}\exp(-2\eta) + b_0 + \exp(2\eta)}. \quad (4.32)$$

Substituting Eq. (4.32) into Eq. (4.2), and by the help of Maple, we have

$$\frac{1}{A}[E_6\exp(6\eta) + E_5\exp(5\eta) + E_4\exp(4\eta) + E_3\exp(3\eta) + E_2\exp(2\eta) + E_1\exp(\eta) + E_0 + E_{-1}\exp(-\eta) + E_{-2}\exp(-2\eta) + E_{-3}\exp(-3\eta) + E_{-4}\exp(-4\eta) + E_{-5}\exp(-5\eta) + E_{-6}\exp(-6\eta)] = 0. \quad (4.33)$$

where

$$A = (b_{-2}\exp(-2\eta) + b_0 + \exp(2\eta))^3,$$

$$E_6 = {a_2}^2 - a_2,$$

$$E_5 = -k^2 a_1 + k a_1 a_2 + \omega a_1 + 2a_1 a_2 - a_1,$$

$$E_4 = 4k^2 a_2 b_0 - 2k{a_2}^2 b_0 - 4k^2 a_0 + 2k a_0 a_2 + k{a_1}^2 - 2\omega a_2 b_0 + {a_2}^2 b_0 + 2\omega a_0 + 2a_0 a_2 + {a_1}^2 - 2a_2 b_0 - a_0,$$

$$E_3 = 6k^2 a_1 b_0 - 3k a_1 a_2 b_0 - 9k^2 a_{-1} + 3k a_{-1} a_2 + 3k a_0 a_1 + 2a_1 a_2 b_0 + 3\omega a_{-1} + 2a_{-1} a_2 + 2a_0 a_1 - 2a_1 b_0 - a_{-1},$$

$$E_2 = -4k^2 a_2 {b_0}^2 + 4k^2 a_0 b_0 + 16k^2 a_2 b_{-2} - 2k a_0 a_2 b_0 - k{a_1}^2 b_0 - 4k{a_2}^2 b_{-2} - 2\omega a_2 {b_0}^2 - 16k^2 a_{-2} + 4k a_{-2} a_2 + 4k a_{-1} a_1 + 2k{a_0}^2 + 2\omega a_0 b_0 - 4\omega a_2 b_{-2} + 2a_0 a_2 b_0 + {a_1}^2 b_0 + {a_2}^2 b_{-2} - a_2 {b_0}^2 + 4\omega a_{-2} + 2a_{-2} a_2 + 2a_{-1} a_1 + {a_0}^2 - 2a_0 b_0 - 2a_2 b_{-2} - a_{-2},$$

$$E_1 = -k^2 a_1 {b_0}^2 - 2k^2 a_{-1} b_0 + 22k^2 a_1 b_{-2} - k a_{-1} a_2 b_0 - k a_0 a_1 b_0 - 7k a_1 a_2 b_{-2} - \omega a_1 {b_0}^2 + 5k a_{-2} a_1 + 5k a_{-1} a_0 + 4\omega a_{-1} b_0 - 2\omega a_1 b_{-2} + 2a_{-1} a_2 b_0 + 2a_0 a_1 b_0 + 2a_1 a_2 b_{-2} - a_1 {b_0}^2 + 2a_{-2} a_1 + 2a_{-1} a_0 - 2a_{-1} b_0 - 2a_1 b_{-2},$$

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$$
\begin{aligned}
E_0 = {} & 6\omega a_{-2}b_0 + 2a_{-2}a_2b_0 - 2a_2b_{-2}b_0 - 12k^2a_{-2}b_0 + 24k^2a_0b_{-2} \\
& + 2a_{-1}a_1b_0 + 6ka_{-2}a_0 - 3ka_1{}^2b_{-2} + 2a_0a_2b_{-2} \\
& + 2a_{-2}a_0 + 3ka_{-1}{}^2 - 2a_0b_{-2} - 2a_{-2}b_0 + a_1{}^2b_{-2} \\
& - 6ka_0a_2b_{-2} - 12k^2a_2b_{-2}b_0 - 6\omega a_2b_{-2}b_0 + a_{-1}{}^2 \\
& + a_0{}^2b_0 - a_0b_0{}^2 , \\
E_{-1} = {} & -k^2a_{-1}b_0{}^2 - 2k^2a_1b_{-2}b_0 + 22k^2a_{-1}b_{-2} + ka_{-2}a_1b_0 \\
& + ka_{-1}a_0b_0 - 5ka_{-1}a_2b_{-2} - 5ka_0a_1b_{-2} + \omega a_{-1}b_0{}^2 \\
& - 4\omega a_1b_{-2}b_0 + 7ka_{-2}a_{-1} + 2\omega a_{-1}b_{-2} + 2a_{-2}a_1b_0 \\
& + 2a_{-1}a_0b_0 + 2a_{-1}a_2b_{-2} - a_{-1}b_0{}^2 + 2a_0a_1b_{-2} \\
& - 2a_1b_{-2}b_0 + 2a_{-2}a_{-1} - 2a_{-1}b_{-2} , \\
E_{-2} = {} & -4k^2a_{-2}b_0{}^2 + 4k^2a_0b_{-2}b_0 - 16k^2a_2b_{-2}{}^2 + 16k^2a_{-2}b_{-2} \\
& + 2ka_{-2}a_0b_0 - 4ka_{-2}a_2b_{-2} + ka_{-1}{}^2b_0 - 4ka_{-1}a_1b_{-2} \\
& - 2ka_0{}^2b_{-2} + 2\omega a_{-2}b_0{}^2 - 2\omega a_0b_{-2}b_0 - 4\omega a_2b_{-2}{}^2 \\
& + 4ka_{-2}{}^2 + 4\omega a_{-2}b_{-2} + 2a_{-2}a_0b_0 + 2a_{-2}a_2b_{-2} \\
& - a_{-2}b_0{}^2 + a_{-1}{}^2b_0 + 2a_{-1}a_1b_{-2} + a_0{}^2b_{-2} - 2a_0b_{-2}b_0 \\
& - a_2b_{-2}{}^2 + a_{-2}{}^2 - 2a_{-2}b_{-2} , \\
E_{-3} = {} & 6k^2a_{-1}b_{-2}b_0 - 9k^2a_1b_{-2}{}^2 + 3ka_{-2}a_{-1}b_0 - 3ka_{-2}a_1b_{-2} \\
& - 3ka_{-1}a_0b_{-2} - 3\omega a_1b_{-2}{}^2 + 2a_{-2}a_{-1}b_0 + 2a_{-1}a_0b_{-2} \\
& - 2a_{-1}b_{-2}b_0 - a_1b_{-2}{}^2 , \\
E_{-4} = {} & 4k^2a_{-2}b_{-2}b_0 - 4k^2a_0b_{-2}{}^2 + 2ka_{-2}{}^2b_0 - 2ka_{-2}a_0b_{-2} \\
& - ka_{-1}{}^2b_{-2} + 2\omega a_{-2}b_{-2}b_0 - 2\omega a_0b_{-2}{}^2 + a_{-2}{}^2b_0 \\
& + 2a_{-2}a_0b_{-2} - 2a_{-2}b_{-2}b_0 + a_{-1}{}^2b_{-2} - a_0b_{-2}{}^2 , \\
E_{-5} = {} & -k^2a_{-1}b_{-2}{}^2 - ka_{-2}a_{-1}b_{-2} - \omega a_{-1}b_{-2}{}^2 + 2a_{-2}a_{-1}b_{-2} \\
& - a_{-1}b_{-2}{}^2 , \\
E_{-6} = {} & a_{-2}{}^2b_{-2} - a_{-2}b_{-2}{}^2 ,
\end{aligned}
$$

Equating the coefficients of $\exp(i\eta)$; $(i = 6, 5, 4, 3, \dots, -6)$ to zero, we have

$$
\begin{cases}
E_6 = 0 , \quad E_3 = 0 , \quad E_0 = 0 , \quad E_{-3} = 0 , \\
E_5 = 0 , \quad E_2 = 0 , \quad E_{-1} = 0 , \quad E_{-4} = 0 , \\
E_4 = 0 , \quad E_1 = 0 , \quad E_{-2} = 0 , \quad E_{-5} = 0 , \\
\qquad\qquad E_{-6} = 0 .
\end{cases} \tag{4.34}
$$

Thus, we obtain a system of algebraic equations for $a_{-2}, a_{-1}, a_0, a_1, a_2, b_{-2}, b_0, \omega$ and k. Solving the system of Eq. (4.34), simultaneously, by the help of Maple, we get

$$
\begin{cases}
a_2 = 1, \quad \omega = -\dfrac{5}{8}, \quad a_0 = a_0, \quad b_0 = b_0 , \quad k = \dfrac{1}{4}, \\
b_{-2} = -a_0{}^2 + a_0b_0, \quad a_1 = a_{-1} = 0, \quad a_{-2} = 0,
\end{cases} \tag{4.35}
$$

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where a_0, b_0 and b_{-2} are free parameters. Let $a_0 = 1$ and $b_0 = -\frac{1}{4}$ therefore, $b_{-2} = -\frac{5}{4}$.

Substituting these results into Eq. (4.32), we obtain the following solitary exact solution

$$u(x,t) = \frac{1 + \exp\left(\frac{1}{2}x - \frac{5}{4}t\right)}{-\frac{5}{4}\exp\left[-\left(\frac{1}{2}x - \frac{5}{4}t\right)\right] - \frac{1}{4} + \exp\left(\frac{1}{2}x - \frac{5}{4}t\right)}. \quad (4.36)$$

which is the solitary wave solution of the nonlinear Burger-Fisher equation when $c = d = 2$ and $p = q = 2$.

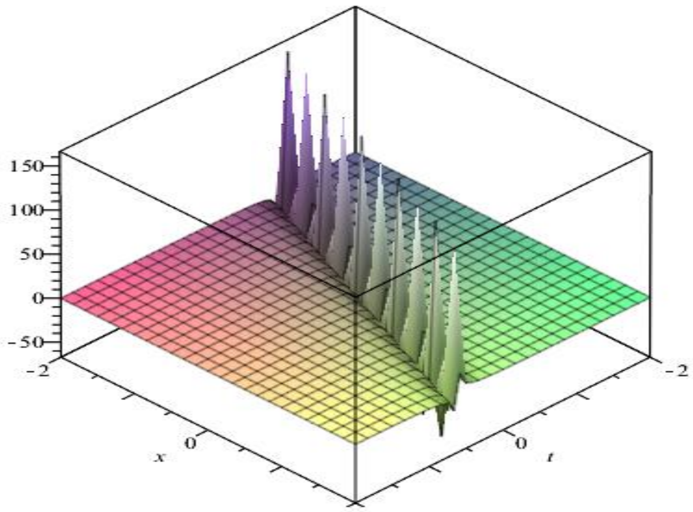

Fig 3: The solitary wave solution when $(b_{-2} = -\frac{5}{4}, k = \frac{1}{4}, \omega = -\frac{5}{8})$

Conclusion

The Exp-function method was successfully to facilitate the process of solving the nonlinear Burger-Fisher equation. Also this method was a straightforward and concise and effective [2], [3], [5].

All final solutions depend on the choice of values c, d, p and q, these values can be freely chosen, but for simplicity, we choose small values [5].

The obtained solutions show that the Exp-function method is promising and powerful mathematical tool for solving various kinds of nonlinear wave equations which arise in mathematical physics, engineering sciences and applied mathematics [8], [13].

The Exp-function method can be generalized for obtaining traveling wave solutions such as solitary solutions, period solutions and compact-like solutions [10], [12].

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