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Some Applications On Class of Uniformly Convex Functions Related by Pre-Schwarzian norm defined by Carlson-Shaffer Operator

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A B S T R A C T

In this paper we obtain norm estimates for the Carlson-Shaffer operator of functions defined on the class of uniformly convex functions of order α and type β by using the Pre-Schwarzian norm given by $\|f\| = \sup_{z\in\mathbb{U}}(1-|z|^2)\left|\frac{f''(z)}{f'(z)}\right|$.

Keywords: Pre-Schwarzian norm, Convex, Starlike, Analytic function, Subordination, Hadamard product, Hypergeometric function, Carlson-Shaffer operator.

بعض التطبيقات علىفصل الدوال المحدبة بشكل منتظم والمرتبطة بنظيم شوارتز والمعرفة باستخدام مؤثر كارلسون–شافير

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الملخص

في هذه الورقة، تحصلنا علي التقديرات المعيارية للمؤثر كارلسون–شافير للدوال المعرفة على فصل الدوال المحدبة بشكل منتظم من الرتبة α والنوع β باستخدام نظيم شوارتز.

الكلمات المفتاحية: نظيم شوارتز،الدالة المحدبة، النجمية، التحليلية، التبيعة التفاضلية، الضرب الالتفافي، الدالة فوق الهندسية، مؤثر كارلسون–شافير.

1. Introduction

Let $\mathcal{H}$ be the space of analytic functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C}: |z| < 1\}$and $\mathcal{H}_a = \{f \in f(0) = a\}$. Additionally, let $\mathcal{A} = \mathcal{H}_1$, meaning that a function $f \in \mathcal{A}$ if

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \,; (z \in \mathbb{U}), \tag{1.1}$$

and let$\mathcal{b}$ be the subclass of $\mathcal{H}$ consisting of all locally univalent functions, namely, $\mathcal{b} = \{f \in \mathcal{H}: f'(z) \neq 0, z \in \mathbb{U}\}$.

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Following Hornich operation [1], we consider $\mathscr{b}$ as a vector space over $\mathbb{C}$, and for $f \in \mathscr{b}$, we define

$$\|f\| = \sup_{z\in\mathbb{U}}(1-|z|^2)\left|\frac{f''(z)}{f'(z)}\right|, \qquad (1.2)$$

where f''/f' is the pre-Schwarzian derivative of f. For general properties of almost uniformly locally univalent functions relating the norm see [2], [3].

For functions $f, g \in \mathcal{H}$, we say that a function f is subordinat to a function g, and written $f \prec g$, if and only if there exists a function ω, analytic in $\mathbb{U}$ such that $\omega(0) = 0$, $|\omega(z)| < 1 (z \in \mathbb{U})$,and $f(z) = g(\omega(z)) (z \in \mathbb{U})$
In particular, if g is univalent in $\mathbb{U}$, then we have the following equivalence (see [4], [5]):
$f(z) \prec g(z) \Leftrightarrow f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$

Also, the Hadamard product (or the convolution) of two analytic functions f and g, where $f(z) = \sum_{n=0}^{\infty} a_n z^n$, and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, is defined as

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n = (g * f)(z), (z \in \mathbb{U})$$

The subclasses of starlike and convex functions of order $\alpha (0 \le \alpha < 1)$ are represented by the notations $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ respectively. We denote $\mathcal{S}^*(0) = \mathcal{S}^*$ and$\mathcal{K}(0) = \mathcal{K}$, the subclasses of starlike and convex functions, see for example (Srivastava and Owa [6]).

The classes UCV and UST of uniformly convex and uniformly starlike functions were first presented by Goodman[7], [8]. Rønning [9] provided a one variable analytic characterization for UCV, meaning that a function $f(z)$ of the form (1.1) belongs to the class UCV if and only if

$$Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf''(z)}{f'(z)}\right|; (z \in \mathbb{U}). \qquad (1.3)$$

Goodman proved the clasical Alexander' s result $f(z) \in UCV \Leftrightarrow zf'(z) \in UST$, does not hold. On later Rønning [10] introduced the class $\mathcal{S}_p$ which consists of functions such that $f(z) \in UCV \Leftrightarrow zf'(z) \in \mathcal{S}_p$. Also, in the follwing Rønning[9], [10] introduce a parameter α to generalized the classes UCV and $\mathcal{S}_p$.

Definition 1.1. A function $f(z)$ of the form (1.1) is said to be in the class $\mathcal{S}_p(\alpha)$, if it satisfies the condition:

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)} - 1\right| + \alpha; (-1 \le \alpha < 1, z \in \mathbb{U}). \qquad (1.4)$$

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The class of uniformly convex functions of order α denoted by $UCV(\alpha)$, is indicated $f(z) \in \mathcal{A}$ such that $zf'(z) \in \mathcal{S}_p(\alpha)$.

Bharati ,Parvatham and Swaminathan also presented the classes $UCV(\alpha,\beta)$ and $\mathcal{S}_p(\alpha,\beta)$ in [11]as follows:

Definition 1.2. A function $f(z)$ of the form (1.1) is said to be in the class $UCV(\alpha,\beta)$, if it satisfies the condition:

$$Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \beta\left|\frac{zf''(z)}{f'(z)}\right| + \alpha; (0 \le \alpha < 1, \beta \ge 0, z \in \mathbb{U}). \qquad (1.5)$$

Definition 1.3. A function $f(z)$ of the form (1.1) is said to be in the class $\mathcal{S}_p(\alpha,\beta)$, if it satisfies the condition:

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \beta\left|\frac{zf'(z)}{f(z)} - 1\right| + \alpha; (0 \le \alpha < 1, \beta \ge 0, z \in \mathbb{U}). \qquad (1.6)$$

Moreover, $f(z) \in UCV(\alpha,\beta)$ if and only if $zf'(z) \in \mathcal{S}_p(\alpha,\beta)$.

For complex parameters $a_1, a_2, \dots, a_q$ and $b_1, b_2, \dots, b_s (b_j \notin \mathbb{Z}_0^- = \{0,-1,-2,\dots\}; j = 1,2,\dots,s)$, the generalized hypergeometric function ${}_qF_s(a_1,\dots,a_q; b_1,\dots,b_s; z)$ is defined by the following series (see [12]):

$${}_qF_s(a_1,\dots,a_q; b_1,\dots,b_s; z) = \sum_{n=2}^{\infty} \frac{(a_1)_n \cdots (a_q)_n}{(b_1)_n \cdots (b_s)_n (1)_n} z^n \qquad (1.7)$$

where$q \le s+1; q,s \in \mathbb{N}_0 = \{0,1,2,\dots\}; z \in \mathbb{U}$and $(\lambda)_n$ is the Pochhammer symbol defined in terms of the Gamma function by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1; & n = 0 \\ \lambda(\lambda+1)(\lambda+2)\dots(\lambda+n-1); & n \in \mathbb{N} \end{cases}$$

Dziok and Srivastava [13], [14] considered a linear operator ${}_qH_s(a_1,\dots,a_q; b_1,\dots,b_s; z): \mathcal{A} \longrightarrow \mathcal{A}$ defined by the following Hadamard product (or convolution):

$${}_qH_s\left(a_1,\dots,a_q; b_1,\dots,b_s; z\right)f(z) = z \ {}_qF_s(a_1,\dots,a_q; b_1,\dots,b_s; z) * f(z)$$
$$= z + \sum_{n=2}^{\infty} \frac{(a_1)_{n-1} \dots (a_q)_{n-1}}{(b_1)_{n-1} \dots (b_s)_{n-1}(1)_{n-1}} a_n z^n \qquad (1.8)$$

For $q = 2$, $s = 1$, $b = 1$ we obtain the Carlson-Shaffer operator defined as (see[6], [15])

$$\mathcal{L}(a,c)f(z) = {}_2H_1(a,1;c,z)f(z)$$
$$(1.9) = z \ {}_2F_1(a,1;c;z) * f(z)$$

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$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^{a-2}(1-t)^{c-a-1} f(tz)dt$$

If $f(z) \in \mathcal{A}$, then we can write (1.9) as follows:

$$\mathcal{L}(a,c)f(z) = z \quad {}_2F_1(a,1;c;z) * f(z)$$

$$= \left(z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} z^n\right) * f(z)$$

$$,= z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} a_n z^n \qquad (1.10)$$

and $z(\mathcal{L}(a,c)f)'(z) = a\mathcal{L}(a+1,c)f(z) - (a-1)\mathcal{L}(a,c)f(z)(z \in \mathbb{U})$.

We notice that

$$\mathcal{L}(a,a)f(z) = f(z), \mathcal{L}(2,1)f(z) = zf'(z)$$

For simplicity, we write ${}_2F_1(a,1;c;z) = F(a,1;c;z)$. For $F(a,1;c;z)$, we have well known derivative formula

$$.F'(a,1;c;z) = \frac{d}{dz}F(a,1;c;z) = \frac{a}{c}F(a+1,2;c+1;z) \qquad (1.11)$$

2. Preliminaries

Each of the following results will be needed to proof our results:

Lemma 2.1. [16]. Let $\psi(z) \in \mathcal{H}_1$ be starlike univalent function with respect to $\psi(0) = 1$, $Re\{\psi(z)\} > 0$, $z \in \mathbb{U}$, and suppose that $g(z) \in \mathcal{A}$ satisfies the equation

$$1 + \frac{zg''(z)}{g'(z)} = \psi(z), (z \in \mathbb{U}).$$

Then for $f \in \mathcal{A}$, the condition $1 + \frac{zg''(z)}{g'(z)} \prec \psi(z)$, implies that $f'(z) \prec g'(z)$.

Lemma 2.2. [17]. Let a, b, c be non zero real numbers with $0 < a \le c$, $a \le 1$.Then the function $zF(a,1;c;z)$ is starlike of order $1 - \frac{a}{2}$.

Corollary 2.1. Suppose that a, b, c be non zero real numbers with $-1 < a \le 1$, $a \le c$. Then the function $F(a,1;c;z)$ is convex.

Lemma 2.3. [18] Let $f, g \in \mathcal{H}$ be a convex function such that $f \prec g$. Then for all convex functions $h \in \mathcal{H}$, we have $h * f \prec h * g$.

Lemma 2.4. [19]. If $f \in \mathcal{K}$, $g \in \mathcal{S}^*$, then for each function h in $\mathcal{A}$, we have $\frac{(f*hg)(\mathbb{U})}{(f*g)(\mathbb{U})} \subseteq \overline{co}h(\mathbb{U})$, where $\overline{co}h(\mathbb{U})$ denotes the closed convex hull of $h(\mathbb{U})$.

3. Results Involving Carlson-Shaffer Operator

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Theorem3.1. [21] Let $f, g \in \mathcal{A}$, g' is convex in $\mathbb{U}$ and $0 < a \leq c$, $a \leq 1$. If $f' \prec g'$, then $\|f\| \leq \|g\|$ and $\|\mathcal{L}(a,c)f(z)\| \leq \|\mathcal{L}(a,c)g(z)\|$.

Proof. Given that $f'(z) \prec g'(z)$, then by applying subordination definition, there exists a ω function analytic in $\mathcal{A}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$, such that
$f'(z) = g'(\omega(z))(z \in \mathbb{U})$.

Additionally, for the Schwarz function ω we have $|\omega(z)| \leq |z|$, that is,

$$(1 - |z|^2)|\omega'(z)| \leq 1 - |\omega(z)|^2, (z \in \mathbb{U}),$$

Thus, we have $\|f\| = sup_{z\in\mathbb{U}}\left[(1 - |z|^2)|\omega'(z)|\left|\frac{g''(\omega(z))}{g'(\omega(z))}\right|\right] \leq sup_{z\in\mathbb{U}}\left[(1 - |\omega(z)|^2)\left|\frac{g''(\omega(z))}{g'(\omega(z))}\right|\right]$

$$= sup_{\xi\in\omega(\mathbb{U})}\left[(1 - |\xi|^2\left|\frac{g''(\xi)}{g'(\xi)}\right|\right] \leq sup_{\xi\in\mathbb{U}}\left[(1 - |\xi|^2\left|\frac{g''(\xi)}{g'(\xi)}\right|\right] = \|g\|,$$

that completes the proof of the first part. To prove the second part, it is enough to show that $\left(\mathcal{L}(a,c)f(z)\right)' \prec \left(\mathcal{L}(a,c)g(z)\right)'$. The function $g'(z)$ is convex (from the theorem) and also $F(a,1;c;z)$ (from Corollary 2.1). Thus, according to Lemma 2.3, we have

$$F(a,1;c;z) * f'(z) \prec F(a,1;c;z) * g'(z),$$

which implies that

$$\left(\mathcal{L}(a,c)f(z)\right)' \prec \left(\mathcal{L}(a,c)g(z)\right)'.$$

Therefore, we complete the proof of Theorem 3.1. □

Theorem 3.2. [21] Let $0 < a \leq c$, $a \leq 1$ and $f(z) \in UCV(\alpha,\beta)(0 \leq \beta < \alpha < 1, \frac{1}{2} \leq \frac{\alpha-\beta}{1-\beta} < 1)$, then for Carlson-Shaffer operator, we have

$$\|\mathcal{L}(a,c)f(z)\| \leq \frac{2a}{c}\left(\frac{1-\alpha}{1-\beta}\right) sup_{0\leq x<1}(1 - x^2)\, \frac{{}_3F_2\left(a+1,2,\frac{3-\beta-2\alpha}{1-\beta};c+1,2;x\right)}{{}_3F_2\left(a,1,\frac{2(1-\alpha)}{1-\beta};c,1;x\right)}.$$

Proof. Given that $f(z) \in UCV(\alpha,\beta)$, then

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta\left|\frac{zf''(z)}{f'(z)}\right| + \alpha,$$

that is,

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \frac{\alpha - \beta}{1 - \beta} \equiv \gamma;\ (z \in \mathbb{U}),$$

or, equivalently,

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{1 + (1 - 2\gamma)z}{1 - z} = \varphi(z); (z \in \mathbb{U}), \tag{3.1}$$

where $\varphi(z)$ is a convex function and therefore starlike with respect $\varphi(z) = 1$. Let $g(z) \in \mathcal{A}$ be such that

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$$1+\frac{zg''(z)}{gf'(z)}=\frac{1+(1-2\gamma)z}{1-z}=\varphi(z);(z\in\mathbb{U}).$$

Then, by Lemma 2.1, we conclude that $f'(z)\prec g'(z)$.
After some computations, we have

$$g'(z)=(1-z)^{2(\gamma-1)}=\sum_{n=o}^{\infty}\frac{(2-2\gamma)_n}{(1)_n}z^n$$
$$=F(2-2\gamma,1;1;z);(z\in\mathbb{U}).$$

Since $\frac{1}{2}\le\gamma<1$, by using Corollary 2.1, the function $g'(z)$ is convex. To find estimation of $\|\mathcal{L}(a,c)f(z)\|$, it is enough from Theorem 2.1 to obtain an estimation of $\|\mathcal{L}(a,c)g(z)\|$. Since

$$(\mathcal{L}(a,c)g)'(z)=F(a,1;c;z)*g'(z)$$
$$=F(a,1;c;z)*F(2-2\gamma,1;1;z)$$

$,=\quad {}_3F_2(a,1,2-2\gamma;c,1;z)$

$$(\mathcal{L}(a,c)g)''(z)=\frac{a(2-2\gamma)}{c}\quad {}_3F_2(a+1,2,3-2\gamma;c+1,2;z)$$

and

$$\left|\frac{{}_3F_2(a+1,2,3-2\gamma;c+1,2;z)}{{}_3F_2(a,1,2-2\gamma;c,1;z)}\right|\le\frac{{}_3F_2(a+1,2,3-2\gamma;c+1,2;|z|)}{{}_3F_2(a,1,2-2\gamma;c,1;|z|)},$$

by Mathematic, we have

$$\left|\frac{(\mathcal{L}(a,c)g)''(z)}{(\mathcal{L}(a,c)g)'(z)}\right|\le\frac{(\mathcal{L}(a,c)g)''(|z|)}{(\mathcal{L}(a,c)g)'(|z|)}(z\in\mathbb{U}).$$

Thus, we conclude that

$$\|\mathcal{L}(a,c)g(z)\|=sup_{z\in\mathbb{U}}(1-|z|^2)\left|\frac{(\mathcal{L}(a,c)g)''(z)}{(\mathcal{L}(a,c)g)'(z)}\right|$$
$$=sup_{z\in\mathbb{U}}(1-|z|^2)\frac{(\mathcal{L}(a,c)g)''(|z|)}{(\mathcal{L}(a,c)g)'(|z|)}$$
$$=\frac{a(2-2\gamma)}{c}sup_{z\in\mathbb{U}}(1-|z|^2)\frac{{}_3F_2(a+1,2,3-2\gamma;c+1,2:|z|)}{{}_3F_2(a,1,2-2\gamma;\mathrm{c},1;|z|)},$$

this completes the proof of Theorem 3.2. □

Theorem 3.3. [20], [21]If $f(z)\in\mathcal{S}^*$, $a>0, c>a+3$ and

$$\frac{\Gamma(c)\Gamma(c-a-1)}{\Gamma(c-a)\Gamma(c-1)}\left[1+\frac{3a}{(c-a-2)}+\frac{2(a)_2}{(c-a-3)_2}\right]\le 2,\qquad(3.2)$$

then $\mathcal{L}(a,c)f(z)\in\mathcal{S}^*$.
Proof. Assuming $f\in\mathcal{S}^*$ i.e.

$$Re\left(\frac{zf'(z)}{f(z)}\right)>0;(z\in\mathbb{U}).\qquad(3.3)$$

Additionally, we have

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$$\frac{z\left(\mathcal{L}(a,c)f(z)\right)'}{\left(\mathcal{L}(a,c)f(z)\right)}=\frac{z\left(zF(a,1;c;z)*f(z)\right)'}{zF(a,1;c;z)*f(z)}=\frac{zF(a,1;c;z)*zf'(z)}{zF(a,1;c;z)*f(z)}$$

$$=\frac{zF(a,1;c;z)*\frac{zf'(z)}{f(z)}f(z)}{zF(a,1;c;z)*f(z)}=\frac{zF(a,1;c;z)*H(z)f(z)}{zF(a,1;c;z)*f(z)}$$

Using the result of [22], we have $zF(a,1;c;z)\in\mathcal{K}$,also $f(z)\in\mathcal{S}^*$ by the hypothesis ofthe theorem. Thus, from Lemma 2.4,

$$\frac{z(\mathcal{L}(a,c)f(z))'}{\mathcal{L}(a,c)f(z)}\in\overline{co}H(\mathbb{U});(z\in\mathbb{U}),$$

and by (3.3), we obtain $\mathcal{L}(a,c)f(z)\in\mathcal{S}^*$, which proves Theorem 3.3. □

Theorem 3.4. [20], [21]Let $a>0, c>a+3$ such that the condition (3.2) holds. If $f\in\mathcal{K}$, then $\mathcal{L}(a,c)f(z)\in\mathcal{K}$.

Proof. Assuming $f\in\mathcal{K}$, it follows that

$$Re\left(1+\frac{zf''(z)}{f'(z)}\right)>0;(z\in\mathbb{U}). \tag{3.4}$$

Furthermore, we have

$$1+\frac{z\left(\mathcal{L}(a,c)f(z)\right)''}{\left(\mathcal{L}(a,c)f(z)\right)'}=1+\frac{F(a,1;c;z)*zf''(z)}{F(a,1;c;z)*f'(z)}$$

$$=\frac{F(a,1;c;z)*(zf''(z)+f'(z))}{F(a,1;c;z)*f'(z)}$$

$$=\frac{zF(a,1;c;z)*\left(1+\frac{zf''(z)}{f'(z)}\right)zf'(z)}{zF(a,1;c;z)*zf'(z)}=\frac{zF(a,1;c;z)*G(z)\{zf'(z)\}}{zF(a,1;c;z)*zf'(z)}$$

Since $zF(a,1;c;z)\in\mathcal{K}$ and $f\in\mathcal{K}$, then using the Alexander theorem, $zf'(z)\in\mathcal{S}^*$ and from Lemma 2.4, we obtain

$$1+\frac{z(\mathcal{L}(a,c)f(z))''}{(\mathcal{L}(a,c)f(z))'}\in\overline{co}G(\mathbb{U});(z\in\mathbb{U}).$$

By (3.4),$\mathcal{L}(a,c)f(z)\in\mathcal{K}$ which ends the proof of Theorem 3.4. □

Theorem 3.5. [20], [21]Let $a>0, c>a+3$ such that the condition (3.2) holds. If f is close to convex with respect to g, then $\mathcal{L}(a,c)f(z)$ is close to convex with respect to $\mathcal{L}(a,c)g(z)$.

Proof. Assuming f is close to convex with respect to g, i.e

$$Re\left(\frac{zf'(z)}{g(z)}\right)>0;(z\in\mathbb{U}). \tag{3.5}$$

Further, we have

$$\frac{z\left(\mathcal{L}(a,c)f(z)\right)'}{\mathcal{L}(a,c)g(z)}=\frac{z\left(zF(a,1;c;z)*f(z)\right)'}{zF(a,1;c;z)*g(z)}=\frac{z(F(a,1;c;z)*f'(z))}{zF(a,1;c;z)*g(z)}$$

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$$= \frac{zF(a,1;c;z) * zf'(z)}{zF(a,1;c;z) * g(z)} = \frac{zF(a,1;c;z) * \frac{zf'(z)}{g(z)} g(z)}{zF(a,1;c;z) * g(z)}$$
$$= \frac{zF(a;1;c;z) * M(z)g(z)}{zF(a,1;c;z) * g(z)}$$

Because of $zF(a,1;c;z) \in \mathcal{K}$ and $g \in \mathcal{S}^*$, Lemma 2.4 gives us

$$\frac{z(\mathcal{L}(a,c)f(z))'}{\mathcal{L}(a,c)g(z)} \in \overline{co}M(\mathbb{U}); (z \in \mathbb{U}),$$

and according to (3.5), we obtain the required result of Theorem 3.5. □

Conclusion

In the present paper, we have obtained the norm estimates for Carlson-Shaffer operator of the class of uniformly convex functions of order α and type β in light of the Pre-Schwarzian norm.

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