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Properties For Generalized Starlike and Convex Functions of Order α

Aisha Ahmed Amer[1] and Maslina Darus[2] and Nagat Muftah Alabbar[3]
[1] Mathmatics Department, Faculty of Science -Al-Khomus, Al-Margib Universit
[2]School of Mathematical Sciences, Faculty of Science and Technology Universiti Kebangsaan Malaysia
[3] Mathmatics Department, Faculty of Education of Benghazi, University of Benghazi

A B S T R A C T
Abstract: The present article focuses on investigating new generalized Starlike and Convex functions of order α that involve the generalized derivative operator .The study aims to analyze various properties of these classes, such as their coefficient inequalities and deduces several intriguing corollaries

Keywords: Analytic functions, Convex functions, Generalized derivative operator, Unit disc.

خصائص الدوال النجمية المعممة والمحدبة من الرتبة α

*عائشة أحمد عامر [1] ، ماسلينا داروس[2]، نجاة مفتاح العبار[3]
[1] قسم الرياضيات، كلية العلوم الخمس جامعة االمرقب,
[2]كلية العلوم الرياضية، كلية العلوم والتكنولوجيا جامعة كيبانجسان ماليزيا
[3] قسم الرياضيات، كلية التربية بنغازي، جامعة بنغازي

الملخص:
تركز هذه المقالة على دراسة الدوال النجمية والمحدبة المعممة الجديدة من الرتبة α التي تتضمن عامل المشتقة المعممة. وتهدف الدراسة إلى تحليل الخصائص المختلفة لهذه الفئات، مثل متباينات معاملاتها واستنتاج العديد من النتائج الطبيعية المثيرة للاهتمام

Introduction

Let A denote the class of all analytic functions in the open unit disc $= \{z: |z| < 1\}$, and satisfy the normalization condition $f(0) = 0, f'(0) = 1$, it has a Taylor series representation

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k .$$

Furthermore, we denote by S the subclass of A consisting of functions of the form above , which are univalent in U.

Starlike functions and convex functions are both classes of analytic functions that have a wide range of applications in complex analysis. These classes of functions are often of interest to many authors and researchers in the field see [2, 15].

Corresponding author: E-mail aisha.amer@elmergib.edu.ly Received 2 febraury2024 - Received in revised form 3 march 2024 Accepted 20 march 2024

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In this paper, we aim to explore and identify certain properties that can be obtained by using generalization starlike and convex functions of order α.

A function f is said to be starlike of order α in the open unit disc U if it is analytic in U and satisfies the condition

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad 0 \le \alpha < 1.$$

A function f is said to be convex of order α in the open unit disc U, if it is analytic in U and satisfies the condition

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \qquad 0 \le \alpha < 1.$$

Now , In [8] the authors introduce the generalized derivative operator $I^m(\lambda_1, \lambda_2, l, n)f(z)$ as the following:

Definition 1.1 If $f \in A$, the generalized derivative operator is defined by

$$I^m(\lambda_1, \lambda_2, l, n): A \longrightarrow A$$

$$I^m(\lambda_1, \lambda_2, l, n)f(z) = \phi^m(\lambda_1, \lambda_2, l)(z) * R^n f(z) \quad , (z \in U)$$

$$I^m(\lambda_1, \lambda_2, L, n)f(z) = z + \sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1)+L)^{m-1}}{(1+L)^{m-1}(1+\lambda_2(k-1))^m} c(n,k) a_k z^k,$$

Where $m \in \mathbb{N}_0 = \{0,1,2, \dots\}$ and $\lambda_2 \ge \lambda_1 \ge 0$, $l \ge 0$, and $R^n f(z)$ denotes the Ruscheweyh derivative operator [3] .

Specical cases of this operator includes :

- The Ruscheweyh derivative operator [3] in the cases:

$$I^1(\lambda_1, 0, l, n) \equiv I^1(\lambda_1, 0,0, n) \equiv I^1(0, 0, l, n) \equiv I^0(0, \lambda_2, 0, n) \equiv I^0(0, 0,0, n)$$

$$\equiv I^{m+1}(0,0, l, n) \equiv I^{m+1}(0, 0,0, n) \equiv R^n$$

- The Salagean derivative operator [4]:

$,I^{m+1}(1, 0,0,0) \equiv S^n$

- The generalized Ruscheweyh derivative operator [5]:

$,I^2(\lambda_1, 0,0, n) \equiv R^n_\lambda$

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- The generalized Salagean derivative operator introduced by Al-Oboudi [11]:

$$I^{m+1}(\lambda_1, 0,0,0) \equiv S_\beta^n \ ,$$

- The generalized Al-Shaqsi and Darus derivative operator [12]:

$$I^{m+1}(\lambda_1, 0,0, n) \equiv R_{\lambda,\beta}^n \ ,$$

- The Al-Abbadi and Darus generalized derivative operator [13] :

$$I^{m}(\lambda_1, \lambda_2, 0, n) \equiv \mu_{\lambda_1,\lambda_2}^{n,m} \ ,$$

And finally

- The Catas derivative operator [14]:

$$.I^{m}(\lambda_1, 0, l, n) \equiv I^{m}(\lambda, \beta, l)$$

We introduce new definitions of generalized Starlike and Convex functions of order α functions containing generalized derivative operator (2) as the following:

Definition 1.2 A function f belonging to A is said to be in the class $S^*(\lambda_1,\lambda_2,L,n,\alpha)$ in U if it satisfies

$$\Re\left\{\frac{z(I^m(\lambda_1,\lambda_2,L,n)f(z))'}{I^m(\lambda_1,\lambda_2,L,n)f(z)}\right\} > \alpha \qquad (z \in U),$$

for some $\alpha(0 \le \alpha < 1)$.

Definition 1.3 A function f belonging to A is said to be in the class $C(\lambda_1,\lambda_2,L,n,\alpha)$ in U if it satisfies

$$\Re\left\{\frac{(z(I^m(\lambda_1,\lambda_2,L,n)f(z))')'}{(I^m(\lambda_1,\lambda_2,L,n)f(z))'}+1\right\} > \alpha \quad (z \in U),$$

for some $\alpha(0 \le \alpha < 1)$.

We note that

$$f(z) \in C(\lambda_1,\lambda_2,L,n,\alpha) \Leftrightarrow zf'(z) \in S^*(\lambda_1,\lambda_2,L,n,\alpha). \tag{3}$$

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Now, for some $\alpha(0<\alpha<1)$, we know that

$$\left|\frac{I^m(\lambda_1,\lambda_2,L,n)f(z)}{z(I^m(\lambda_1,\lambda_2,L,n)f(z))'}-\frac{1}{2\alpha}\right|<\frac{1}{2\alpha}\Leftrightarrow\Re\left\{\frac{z(I^m(\lambda_1,\lambda_2,L,n)f(z))'}{I^m(\lambda_1,\lambda_2,L,n)f(z)}\right\}>\alpha.$$

Then, it is easy to show that

$$\left|\frac{I^m(\lambda_1,\lambda_2,L,n)f(z)}{z(I^m(\lambda_1,\lambda_2,L,n)f(z))'}-\frac{1}{2\alpha}\right|<\frac{1}{2|\alpha|}\Leftrightarrow\Re\left\{\frac{1}{\alpha}\frac{z(I^m(\lambda_1,\lambda_2,L,n)f(z))'}{I^m(\lambda_1,\lambda_2,L,n)f(z)}\right\}>1.$$

for some α such that $|\alpha-\frac{1}{2}|<\frac{1}{2}$.

2 Coefficient inequalities

We consider our coefficient inequalities for $f\in S_\alpha(\lambda_1,\lambda_2,L,n)$ and $f\in C_\alpha(\lambda_1,\lambda_2,L,n)$.

Theorem 2.1 If a function $f\in A$ satisfies the following inequality

$$\sum_{k=2}^{\infty}(|\frac{(1+\lambda_1(k-1)+L)^{m-1}}{(1+L)^{m-1}(1+\lambda_2(k-1))^m}c(n,k)(2\alpha-k)|$$
$$+k\frac{(1+\lambda_1(k-1)+L)^{m-1}}{(1+L)^{m-1}(1+\lambda_2(k-1))^m}c(n,k))|a_k|\le 1-|1-2\alpha|,$$

for some complex number α such that $|\alpha-\frac{1}{2}|<\frac{1}{2}$, then $f\in S_\alpha(\lambda_1,\lambda_2,L,n)$.

Proof: We know that

$$\left|\frac{I^m(\lambda_1,\lambda_2,L,n)f(z)}{z(I^m(\lambda_1,\lambda_2,L,n)f(z))'}-\frac{1}{2\alpha}\right|<\frac{1}{2|\alpha|}\Leftrightarrow\Re\left\{\frac{1}{\alpha}\frac{z(I^m(\lambda_1,\lambda_2,L,n)f(z))'}{I^m(\lambda_1,\lambda_2,L,n)f(z)}\right\}>1,$$

for some complex number α such that $|\alpha-\frac{1}{2}|<\frac{1}{2}$. Since

$$\left|\frac{I^m(\lambda_1,\lambda_2,L,n)f(z)}{z(I^m(\lambda_1,\lambda_2,L,n)f(z))'}-\frac{1}{2\alpha}\right|$$

$$\le\frac{1}{2|\alpha|}\frac{|2\alpha-1|+\sum_{k=2}^{\infty}|\frac{(1+\lambda_1(k-1)+L)^{m-1}}{(1+L)^{m-1}(1+\lambda_2(k-1))^m}c(n,k)(2\alpha-k)||a_k||z|^{k-1}}{1-\sum_{k=2}^{\infty}k\frac{(1+\lambda_1(k-1)+L)^{m-1}}{(1+L)^{m-1}(1+\lambda_2(k-1))^m}c(n,k)|a_k||z^{k-1}|}$$

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$$< \frac{1}{2|\alpha|}\frac{|2\alpha-1|+\sum_{k=2}^{\infty}|\frac{(1+\lambda_1(k-1)+L)^{m-1}}{(1+L)^{m-1}(1+\lambda_2(k-1))^m}c(n,k)(2\alpha-k)||a_k|}{1-\sum_{k=2}^{\infty}k\frac{(1+\lambda_1(k-1)+L)^{m-1}}{(1+L)^{m-1}(1+\lambda_2(k-1))^m}c(n,k)|a_k|},$$

if f satisfies the following equality

$$|2\alpha-1|+\sum_{k=2}^{\infty}\left|\frac{(1+\lambda_1(k-1)+L)^{m-1}}{(1+L)^{m-1}(1+\lambda_2(k-1))^m}c(n,k)(2\alpha-k)\right||a_k|$$

$$\le 1-\sum_{k=2}^{\infty}k\frac{(1+\lambda_1(k-1)+L)^{m-1}}{(1+L)^{m-1}(1+\lambda_2(k-1))^m}c(n,k)|a_k|,$$

that is,

$$\sum_{k=2}^{\infty}\left(\left|\frac{(1+\lambda_1(k-1)+L)^{m-1}}{(1+L)^{m-1}(1+\lambda_2(k-1))^m}c(n,k)(2\alpha-k)\right|\right.$$

$$\left.+k\frac{(1+\lambda_1(k-1)+L)^{m-1}}{(1+L)^{m-1}(1+\lambda_2(k-1))^m}c(n,k)\right)|a_k|\le 1-|1-2\alpha|,$$

we obtain that

$$\left|\frac{I^m(\lambda_1,\lambda_2,L,n)f(z)}{z(I^m(\lambda_1,\lambda_2,L,n)f(z))'}-\frac{1}{2\alpha}\right|<\frac{1}{2|\alpha|}\quad (z\in \mathrm{U}).$$

The proof is complete.

Making $\alpha\in R$ in Theorem 2.1,we obtain the next corollary.

Corollary 2.1 If a function $f\in A$ satisfies the following inequality

$$\sum_{k=2}^{\infty}(k-\alpha)(\frac{(1+\lambda_1(k-1)+L)^{m-1}}{(1+L)^{m-1}(1+\lambda_2(k-1))^m}c(n,k))|a_k|\le\begin{cases}\alpha & if\quad (0<\alpha\le\frac{1}{2}),\\ 1-\alpha & if\quad (\frac{1}{2}<\alpha<1),\end{cases}$$

for some real $\alpha(0<\alpha<1)$, then $f\in S^*(\lambda_1,\lambda_2,L,n,\alpha)$.

Corollary 2.2 If a function $f\in A$ and $\lambda_2=0, m=1$,then

$$\sum_{k=2}^{\infty}(k-\alpha)|a_k|\le\begin{cases}\alpha & if\quad (0<\alpha\le\frac{1}{2}),\\ 1-\alpha & if\quad (\frac{1}{2}<\alpha<1),\end{cases}$$

for some real $\alpha(0<\alpha<1)$, then $f\in S^*(\alpha)$, see [6].

Theorem 2.2 If a function $f\in A$ satisfies the following inequality

$$\sum_{k=2}^{\infty}k(|\frac{(1+\lambda_1(k-1)+L)^{m-1}}{(1+L)^{m-1}(1+\lambda_2(k-1))^m}c(n,k)(2\alpha-k)|$$

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$$+k\,\frac{(1+\lambda_1(k-1)+L)^{m-1}}{(1+L)^{m-1}(1+\lambda_2(k-1))^{m}}c(n,k))|a_k|\le 1-|1-2\alpha|,$$

for some complex number α such that $|\alpha-\frac{1}{2}|<\frac{1}{2}$, then $f\in C_\alpha(\lambda_1,\lambda_2,L,n)$.

Proof: Since, $f\in C_\alpha(\lambda_1,\lambda_2,L,n)$ if and only if $zf'(z)\in S_\alpha(\lambda_1,\lambda_2,L,n)$, and we may replace a_n with na_n in Theorem 2.1 .

Corollary 2.3 If a function $f\in A$ and $\lambda_2=0, m=1$,then

$$\sum_{k=2}^{\infty}k\,(|2\alpha-k|+k)|a_k|\le 1-|1-2\alpha|,$$

for some complex number α such that $|\alpha-\frac{1}{2}|<\frac{1}{2}$, then $f\in C_\alpha$,see [6].

These references [1,7,9] contain additional research and studies on analytic functions connected to the derivative operator and integral operator. It is advisable to consult these sources for further information and insights.

Conclusion

We have examined this operator mentioned within the classes of analytic functions, specifically focusing on its applications in geometric function theory.
The suggested operator can be employed in order to establish additional categories of analytic functions, or to extend the applicability of other types of differential operators.

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