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Special Properties of Differential Subordination Functions in the Classes of Analytic Functions

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A B S T R A C T

This work presents a novel investigation a generalized derivative operator which is a multivalent function by using Hadamard product in the field of geometric function theory. Also, we explain the use of the convexity and the starlikeness properties of a given function to consequences of superordination and differential subordination with a specific focus on sandwich theorems.

Keywords: Analytic function - univalent function - differential subordination - generalized derivative operator - sandwich theorem

الخصائص الخاصة لدوال التبعية التفاضلية في فئات الدوال التحليلية

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الملخص

يقدم هذا العمل بحثا جديدا عن عامل المشتقة المعممة وهي دالة متعددة التكافؤ باستخدام منتج هادامارد في مجال نظرية الوظيفة الهندسية. كما قمنا بشرح استخدام خصائص التحدب والتشابه النجمي لدالة معينة لتفوق والتبعية التفاضلية مع التركيز بشكل خاص على نظريات الساندويتش. في فئات الوظائف التحليلية

الكلمات المفتاحية: الدوال التحليلية، الدوال احادية التكافؤ،العامل المشتق المعمم, الدالة التبعية التفاضلية, نظرية الساندويتش.

Introduction

In the past, people have used complex numbers to solve real cubic equations, which has facilitated the development of a fascinating theory known as the theory of functions of a complex variable (complex analysis). This field has a historical origin dating back to the 17th century. Noteworthy figures in the field include Riemann, Gauss, Euler, Cauchy, Mittag-Leffler, and several more scientists. Riemann introduced the Riemann mapping theorem in 1851 during the 19th century, giving rise to geometric function theory (GFT), a notable and captivating theoretical framework [11]. It has seen significant development and has been applied in several scientific domains, including operator theory, differential inequality theory, and other related topics. To enhance the Riemann mapping theorem, Koebe [11] utilized a univalent function defined on an open unit disk in 1907.

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In 1909, Lindeof introduced the subordinate idea. The Schwarz function is employed to examine two complex functions. Diverse subordination theory on a complex domain may be understood as an extension of differential inequality theory on a real domain.

Miller and Mocanu have investigated the theory of first and second order differential subordination. Recently Miller and Mocanu investigated the dual concept of differential superordination to obtain several sandwich results, see [21,20,13].

The application of the subordination technique is employed in relation to pertinent categories of permissible functions. According to Antonino and Miller [7], the acceptable functions are defined as follows:

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form :

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in \mathbb{U}) \ ,$$

Which are analytic in the open unit disk $\mathbb{U} = \{z: |z| < 1; z \in \mathbb{C}\}$ where $a_k$is a complex number .

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of univalent functions in $\mathbb{U}$ and by C the familiar subclass of $\mathcal{S}$ whose members are convex functions.

If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ *and* $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$; $(z \in \mathbb{U})$, then the Hadamard product of two analytic functions $f$and g denoted by $f * g$ is defined by

$$(f * g)(z) = f(z) * g(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k \quad , \quad (z \in \mathbb{U}).$$

And by using this product, the authors in [4] have recently introduced a new generalized derivative operator given by:

Definition 1.

The shifted factorial $(c)_k$ can be defined as:

$$(c)_k = c(c+1) \ldots..(c+k-1) \ \ if \ k \in \mathbb{N} = \{1,2,3,\ldots\}, c \in \mathbb{C} - \{0\}.$$

and

$$(c)_k = 1 \ \ if \ k = 0.$$

Definition 2:

The $(c)_k$ can be expressed in terms of the Gamma function as :

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$$(c)_k = \frac{\Gamma(c+k)}{\Gamma(c)} \ ,(n \in \mathbb{N}).$$

In order to derive the generalized derivative operator [4], we define the analytic function

$$\phi^m(\lambda_1,\lambda_2,l)(z) = z + \sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m} z^k, \qquad (1)$$

Where $m \in \mathbb{N}_0 = \{0,1,2, \ldots.\}$ and $\lambda_2 , \lambda_1, l \in \mathbb{R}$ such that $\lambda_2 \geq \lambda_1 \geq 0 , l \geq 0$.

Now , In [11] the authors introduce the generalized derivative operator $I^m(\lambda_1,\lambda_2,l,n)f(z)$

As the following :

Definition 3.

For $f \in \mathcal{A}$ the operator $I^m(\lambda_1,\lambda_2,l,n)$ is defined by $I^m(\lambda_1,\lambda_2,l,n): A \longrightarrow A$

$$I^m(\lambda_1,\lambda_2,l,n)f(z) = \phi^m(\lambda_1,\lambda_2,l)(z) * R^n f(z) \ \ ,(z \in \mathcal{D}) \qquad (2)$$

Where $m \in \mathbb{N}_0 = \{0,1,2, \ldots.\}$ and $\lambda_2 \geq \lambda_1 \geq 0 , l \geq 0$, and $R^n f(z)$ denotes the Ruscheweyh derivative operator [3] ,and given by

$$R^n f(z) = z + \sum_{k=2}^{\infty} c(n,k) a_k z^k \, ,(n \in \mathbb{N}_0, z \in \mathcal{D}),$$

Where

$$(c)_k = \frac{\Gamma(c+k)}{\Gamma(c)} \ .$$

If f is given by (1) , then we easily find from the equality (3) that

$$I^m(\lambda_1,\lambda_2,l,n)f(z) = z + \sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}\big(1+\lambda_2(k-1)\big)^m} c(n,k) a_k z^k,$$

Where $n, m \in \mathbb{N}_0 = \{0,1,2, \ldots.\}$, $\lambda_2 \geq \lambda_1 \geq 0 , l \geq 0$, $(c)_k = \frac{\Gamma(c+k)}{\Gamma(c)}$.

Special cases of this operator includes:

- The Ruscheweyh derivative operator [15] in the cases :
$$I^1(\lambda_1,0,l,n) \equiv I^1(\lambda_1,0,0,n) \equiv I^1(0,0,l,n) \equiv I^0(0,\lambda_2,0,n) \equiv I^0(0,0,0,n) \equiv I^{m+1}(0,0,l,n) \equiv I^{m+1}(0,0,0,n) \equiv R^n.$$

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- The Salagean derivative operator [16] :

$$I^{m+1}(1,0,0,0) \equiv D^n.$$

- The generalized Ruscheweyh derivative operator[15] :

$$I^2(\lambda_1, 0,0, n) \equiv R_\lambda^n.$$

- The generalized Salagean derivative operator introduced by [2] :

$$I^{m+1}(\lambda_1, 0,0,0) \equiv D_\beta^n.$$

- The generalized Al-Shaqsi and Darus derivative operator [17] :

$$I^{m+1}(\lambda_1, 0,0, n) \equiv D_{\lambda,\beta}^n .$$

- The Al-Abbadi and Darus generalized derivative operator [3] :

$$I^m(\lambda_1, \lambda_2, 0, n) \equiv \mu_{\lambda_1,\lambda_2}^{n,m} .$$

- And finally the catas derivative operator [8] :

$$I^m(\lambda_1, 0, l, n) \equiv I^m(\lambda_1, \beta, l).$$

Using simple computation one obtains the next result

$$(\ell+1)I^{m+1}(\lambda_1,\lambda_{2,}\ell,n)f(z) = (1+\ell-\lambda_1)[I^m(\lambda_1,\lambda_{2,}\ell,n) * \varphi^1(\lambda_1,\lambda_2,\ell)(z)]f(z) + \lambda_1 z\left[I^m\left(\lambda_1,\lambda_{2,}\ell,n\right) * \varphi^1(\lambda_1,\lambda_2,\ell)(z)\right]', \qquad (*)$$

where (z ∈ 𝕌) and $\varphi^1(\lambda_1, \lambda_2, \ell)(z)$ analytic function given by

$$\phi^1(\lambda_1,\lambda_2,l)(z) = z + \sum_{k=2}^{\infty} \frac{1}{(1+\lambda_2(k-1))} z^k .$$

Now, we assume that the function f *and* g are analytic in 𝕌 ,then we say f is subordinate to g or g is said to be superordinate to f in 𝕌 ,written as $f \prec g$ *or* $f(z) \prec g(z)$ if there is a Schwarz function $v(z)$ analytic in 𝕌 , *with* $|v(z)| < 1$, So that $f(z) = g\left(v(z)\right)$; $z \in$ 𝕌 .

In particular, If the function g is univalent in 𝕌 then the subordination $f \prec g$ is equivalent to $f(0) = g(0)$ *and* $f(\mathbb{U}) = g(\mathbb{U})$.

Definition 4:

Let ℂ3 × 𝕌 → ℂ and let ћ (z) is univalent in 𝕌. If $f(z)$ is analytic function in 𝕌 and satisfies the second-order differential subordination:

$$\psi\ (p(z), zp'(z), z^2\ p''(z); z) \prec ћ(z) , \qquad (3)$$

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Then p(z) is said to be a solution of the differential subordination (3). The solutions of equation (3) of differential subordination have dominant univalent function q(z) or more simply a dominant, if $p(z) \prec$ q(z) to all $p(z)$ satisfying (3).
A dominant function $\tilde{q}$ (z) that satisfies $\tilde{q}$ (z) $\prec$ q(z) for all dominant q(z) of (3) is called the best dominant of (3).

Definition 5:
Let ψ: ℂ3 × 𝕌 ⇢ ℂ and let ħ(z) be analytic function in 𝕌. If ((z), $zp'(z)$, z^2 $p''(z)$; z) *and* $p(z)$ are univalent functions in 𝕌 and $p(z)$ satisfies the second-order differential superordination :

$$\hbar\,(z) \prec \psi(p(z), zp'(z), z^2p''(z); z)\,, \qquad (4)$$

then $p(z)$ is said to be a solution of the differential superordination (4). The analytic function q(z) is said to be a subordinant of the solutions of equation (4) of the differential superordination, or more simply a subordinate, if q(z) $\prec p(z)$ for all $p(z)$ satisfying (4).

If q(z) $\prec\tilde{q}$ (z) for all subordinates q(z) of (4) which is satisfied by univalent subordinate $\tilde{q}$ (z), then q(z) is said to be the best subordinate . Ali et al. [18,19] get sufficient consideration for normalize analytic functions to hold.

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z)$$

such that $q_1(z)$ and $q_2(z)$ represent univalent normalized functions in 𝕌 that take the form of univalent function with

$$q_1(0) = q_2(0) = 1$$

The major objective of present implementing is to discover enough conditions to a certain normalize analytic functions f to give:

$$q_1(z) \prec \frac{z\left(I^m\left(\lambda_1, \lambda_{2,}\ell, n\right)f(z)\right)'}{I^m(\lambda_1, \lambda_{2,}\ell, n)f(z)} \prec q_2(z)$$

such that $q_1(z)$ and $q_2(z)$in 𝕌 are called univalent functions with

$$q_1(0) = q_2(0) = 1.$$

2. Preliminaries:

Definition 6 : [7]

Let Q be the set to all functions f that are analytic and injective on $\overline{\mathbb{U}}\backslash E(f)$, such that

$$\overline{\mathbb{U}} = \mathbb{U} \cup (z \in \partial U)\ \text{, and } E(f) = \{\zeta \in \partial\mathbb{U}: \lim_{z\to\zeta} f(z) = \infty\}, \qquad (5)$$

and $f'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U}\backslash E(f)$. The subclass of Q for which $f(0) = a$ is denoted by $Q(a)$, where $Q(1) = Q_1$ *and* $Q(0) = Q_0$.

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Lemma (1): [21]:

Assume θ *and* φ are analytic function in a domain D involving $q(u)$ with $\varphi(w) \neq 0$ such that $w \in q(u)$. Take $h(z) = \theta(q(z)) + \psi(z)$ *and* $\psi(z) = zq'(z)\varphi(q(z))$.

Furthermore, we assume that

(1) $\psi(z)$ be univalent starlike function in $\mathbb{U}$,

for $z \in \mathbb{U}$. (2) $Re\left\{\frac{zh'(z)}{\psi(z)}\right\} > 0$

If $p(z)$ *is* analytic in $\mathbb{U}$ with $p(u) \subseteq D, p(0) = q(0)$ *and*

$$(6) \quad \theta(p(z)) + zp'(z)\varphi(q(z)) \prec \theta(q(z)) + zp'(z)\varphi(q(z)).$$

Then q to be the best dominant and $p \prec q$.

Lemma (2): [7] Assume that

$$\mathrm{Re}\{1 + \frac{zq''(z)}{q'(z)}\} > \max\left\{0, -Re\left(\frac{\alpha}{\beta}\right)\right\}$$

be univalent convex function in $\mathbb{U}$ and suppose $\alpha \in \mathbb{C}$, $\beta \in \frac{\mathbb{C}}{\{0\}}$ such that*and let* $q(z)$

$$(7)\alpha p(z) + \beta zp'(z) \prec \alpha q(z) + \beta zq'(z),$$

Then $p(z) \prec q(z)$*and* $q(z)$*will be the best subordinanat*.

Lemma (3): [13]

Assume $q(z)$ *is* univalent convex function in $\mathbb{U}$ *and* $q(0) = 1$. Suppose that $\beta \in \mathbb{C}$ such that Re $(\beta) > 0$. If $p(z) + \beta zp'(z)$ in $\mathbb{U}$ is univalent and $p(z) \in H[q(0), 1] \cap Q$, *satisfies* $q(z) + \beta zq'(z) \prec p(z) + \beta zp'(z)$, *then* $q(z) \prec p(z)$ and $q(z)$*is the best subordinant*.

3. Subordination Results:

Theorem (1)

Assume $q(z)$ is univalent convex function in $\mathbb{U}$ with $q(z) \neq 0$, and $q(0) = 1$. If $q(z)$ satisfies:

$$Re\{1 + \frac{zq''(z)}{q'(z)} > \max\left\{0, Re\left(\frac{1}{\vartheta}\right)\right\}, for\ all\ z \in \mathbb{U}, \qquad (8)$$

where $\vartheta \in \mathbb{C}^* = \mathbb{C} - \{0\}$. We also assume

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$$\psi(z) = \left(1 - \frac{\vartheta}{X}\right)\left(\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z}\right) + \frac{\vartheta}{X}\left(\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z}\right), \text{ x} > 0. \quad (9)$$

If $q(z)$ satisfies the subordination

$$\psi(z) \prec q(z) + \vartheta z q'(z), \qquad (10)$$

Then

$$\left(\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z}\right) \prec q(z), \qquad (11)$$

and $q(z)$ will be the best dominant of equation (10).

Proof:

Let

$$p(z) = \left(\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z}\right), z \in \mathbb{U}. \qquad (12)$$

By taking the differentiation of (12) logarithmically with respect to z, then we obtain

$$\frac{zp'(z)}{p(z)} = \frac{z(I^m(\lambda_1,\lambda_2,l,n)f(z))'}{I^m(\lambda_1,\lambda_2,l,n)f(z)} - 1. \qquad (13)$$

Now, we use the identity (*) in (13), then we get

$$\frac{zp'(z)}{p(z)} = \frac{1}{\sigma}\left(\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{I^m(\lambda_1,\lambda_2,l,n)f(z)} - 1\right).$$

Therefore, we apply Lemma (2) with $\alpha = 1$ and $\beta = \vartheta$,

implies that $q(z)$ is best dominant and $p(z) < q(z)$.so that the subordination (10)

In this step, we set $-1 \le B \le A < 1$, and $q(z) = \frac{1+Az}{1+Bz}$, in previous theorem.

The condition (8) becomes

$$Re\left\{\frac{1-Bz}{1+Bz}\right\} > \max\left\{0, -Re\left(\frac{1}{\vartheta}\right)\right\}, z \in \mathbb{U}. \qquad (14)$$

The function $\varphi(\partial) = \frac{1-\partial}{1+\partial}, |\partial| < |B|$ is a convex function in $\mathbb{U}$ and because of $\varphi(\bar{\partial}) = \overline{\varphi(\partial)}$ for all $|\partial| < |B|$, there for the image $\varphi(\mathbb{U})$ will be a convex domain symmetrically according to real axis, as a result,

$$\inf\left\{ \text{Re}\left(\frac{1-Bz}{1+Bz}\right) : z \in \mathbb{U} \right\} = \frac{1-|B|}{1+|B|} > 0.$$

The inequality (14) is equivalent to

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$$\text{Re}\{\frac{1}{\vartheta}\} > \frac{|B|-1}{1+|B|}.$$

Therefore, we get the result corollary.

Corollary (1)

Assume that $\max\left\{0, -Re\left(\frac{1}{\vartheta}\right)\right\} \leq \frac{1-|B|}{1+|B|}\ and -1 \leq B \leq A < 1.$

If $\psi(z) \prec \frac{1-Az}{1+Bz} + \vartheta\frac{A-B}{(1+Bz)^2}$, then

$$) \prec \frac{1+Az}{1+Bz}, (\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z}$$

and the term of $\frac{1+Az}{1+Bz}$ will be the best dominant.

Corollary (2):

Suppose that

$$Re\left(\frac{1}{\alpha}\right) \geq 0, if\ \psi(z) \prec \frac{1+z}{1-z} + \vartheta\frac{2z}{(1+z)^2}.$$

Then Re $(\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z}) > 0$, and the term of $\frac{1+z}{1-z}$ will be the best dominant.

Now, for $q(z) = e^{Az}, |A| < \pi.$

The next corollary is obtained by theorem (3.1).

Corollary (3):

Assume $\boldsymbol{Re(1+Az)} > \mathbf{max}\left\{\mathbf{0}, -\boldsymbol{Re}\left(\frac{\mathbf{1}}{\boldsymbol{\vartheta}}\right)\right\}, |\boldsymbol{A}| < \boldsymbol{\pi}, \boldsymbol{such\ that\ \vartheta} \in \mathbb{C}^*, \boldsymbol{\psi(z)} \prec (\mathbf{1} + \boldsymbol{\vartheta zA})\boldsymbol{e^{Az}}, \boldsymbol{then\ Re}(\frac{\boldsymbol{I^m(\lambda_1,\lambda_2,l,n)f(z)}}{\boldsymbol{z}}) < \boldsymbol{e^{Az}}, \boldsymbol{and\ e^{Az}}$ will be the best dominant.

Theorem (2):

Let q(z) be univalent function in $\mathbb{U}$, such that $q(z) \neq 0 and\ q(0) = 1$ for $z \in \mathbb{U}$ and suppose that q(z) satisfies the term of $Re\left(\frac{zq'(z)}{q(z)}\right) > 0$, which is univalent and starlike function in $\mathbb{U}$.

Now, let $\gamma_1, \gamma_2, \alpha, \partial \in \mathbb{C}^*, with\ \gamma_1 + \gamma_2 \neq 0,$

$$(15)\frac{\gamma_1 I^m(\lambda_1,\lambda_2,l,n)f(z) + \gamma_2 I^m(\lambda_1,\lambda_2,l,n)f(z)}{(\gamma_1+\gamma_2)z} \neq 0, z \in \mathbb{U}.$$

And

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$$[1+\alpha\partial\left(\frac{\gamma_1 z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'+\gamma_2 z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{\gamma_1 I^m(\lambda_1,\lambda_2,l,n)f(z)+\gamma_2 I^m(\lambda_1,\lambda_2,l,n)f(z)}-1\right) \prec 1+\partial\frac{zq'(z)}{q(z)}], \qquad (16)$$

Then q(z) will be the best dominant of equation (16) , and
$(\frac{\gamma_1 I^m(\lambda_1,\lambda_2,l,n)f(z)+\gamma_2 I^m(\lambda_1,\lambda_2,l,n)f(z)}{(\gamma_1+\gamma_2)^z})^\alpha \prec q(z).$

Proof: Suppose that

$$(17) p(z) = (\frac{\gamma_1 I^m(\lambda_1,\lambda_2,l,n)f(z)+\gamma_2 I^m(\lambda_1,\lambda_2,l,n)f(z)}{(\gamma_1+\gamma_2)^z})^\alpha, z \in \mathbb{U}.$$

According to (15), we have p(z) is analytic function in $\mathbb{U}$.

By taking the differentiation of (17) logarithmically according to z, we obtain

$$\frac{zp'(z)}{p(z)} = \alpha\left(\frac{\gamma_1 z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'+\gamma_2 z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{\gamma_1 I^m(\lambda_1,\lambda_2,l,n)f(z)+\gamma_2 I^m(\lambda_1,\lambda_2,l,n)f(z)}-1\right).$$

To prove our result , we use lemma (2).

Now, consider $(w) = \frac{\vartheta}{w}$ $and\ \theta(w) = 1$,we note that $w \in \mathbb{C}^*, and\ \theta$ is analytic function in $\mathbb{C}$

We also suppose $h(z) = \theta(q(z)) + \psi(z) = 1 + \partial\frac{zq\prime(z)}{q(z)}$ and

so that the function $\psi(z)$ is starlike in $\mathbb{U}$, and $\psi(z) = zq'(z)\varphi(q(z)) = \partial\frac{zq\prime(z)}{q(z)}$,

$$Re\left(\frac{zh'(z)}{\psi(z)}\right) = Re(1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}) > 0.$$

Therefore, the subordination (16) implies that q(z) is the best dominant and $p(z) \prec q(z)$.

The next result can be obtained by setting $-1 \le B \le A < 1, q(z) = \frac{1+Az}{1+Bz}, \gamma_2 = 0\quad and\ \partial = 1\ in\ theorem$ (3).

Corollary (4):

Suppose that $-1 \le B \le A < 1$ and $\left(\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z}\right) \neq 0, z \in \mathbb{U}, \alpha \in \mathbb{C}^*$.

If $[1+\alpha\left(\frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)}-1\right) \prec 1+\frac{(A-B)z}{(1+Az)(1-Bz)}], then\ (\frac{Az+1}{Bz+1})$ will be best dominant and $(\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z})^\alpha \prec \frac{Az+1}{Bz+1}$.

Theorem (3):

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Suppose *that the univalent function q(z) in* $\mathbb{U}$ *with q(z)*$\neq 0, q(0) = 1, \forall z \in \mathbb{U}, \alpha, \partial \in \mathbb{C}^*, \gamma_1\gamma_2, \varphi, \tau \in \mathbb{C}, with\ \gamma_1 + \gamma_2 \neq 0$ and (15) is satisfied.

Let $Re(1 + \frac{zq"(z)}{q'(z)}) > \max\{0, -Re\left(\frac{\varphi}{\partial}\right)\}$, and

$$\Theta(z) = (\frac{\gamma_1 I^m(\lambda_1,\lambda_2,l,n)f(z) + \gamma_2 I^m(\lambda_1,\lambda_2,l,n)f(z)}{(\gamma_1+\gamma_2)z})^\alpha \times$$

$$[\varphi + \alpha\partial(\left(\frac{\gamma_1 z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)' + \gamma_2 z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{\gamma_1 I^m(\lambda_1,\lambda_2,l,n)f(z) + \gamma_2 I^m(\lambda_1,\lambda_2,l,n)f(z)} - 1\right)] + \tau. \qquad (18)$$

If q(z) holds the subordination

$$(19)\Theta(z) \prec \varphi q(z) + \partial z q'(z) + \tau,$$

then $(\frac{\gamma_1 I^m(\lambda_1,\lambda_2,l,n)f(z)+\gamma_2 I^m(\lambda_1,\lambda_2,l,n)f(z)}{(\gamma_1+\gamma_2)z})^\alpha \prec q(z),$ (20)

and q(z) will be the best dominant of equation (19).

Proof: Assume

$$(21)\xi(z) = (\frac{\gamma_1 I^m(\lambda_1,\lambda_2,l,n)f(z)+\gamma_2 I^m(\lambda_1,\lambda_2,l,n)f(z)}{(\gamma_1+\gamma_2)z})^\alpha,$$

then the function $\xi(z)$ be analytic in $\mathbb{U}$ and q(0)=1, hence by taking the differentiation of equation. (21) logarithmically with respect to z, and by taking the (*) in a recent equation,

$$\Theta(z) = (\frac{\gamma_1 I^m(\lambda_1,\lambda_2,l,n)f(z) + \gamma_2 I^m(\lambda_1,\lambda_2,l,n)f(z)}{(\gamma_1+\gamma_2)z})^\alpha \times [\varphi + \alpha\partial(\left(\frac{\gamma_1 z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)' + \gamma_2 z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{\gamma_1 I^m(\lambda_1,\lambda_2,l,n)f(z) + \gamma_2 I^m(\lambda_1,\lambda_2,l,n)f(z)} - 1\right)] + \tau.$$

Thus the subordination (20) is against to

$$\xi\varphi(z) + \partial z\xi'(z) + \tau \prec \varphi q(z) + \partial z q'(z) + \tau.$$

Now, if we apply of lemma (2) with $\beta = \frac{\partial}{\varphi}$ and $\alpha = 1$, then we have (20).

We also get the next corollary, if we substitute $-1 \leq \mathrm{B} \leq \mathrm{A} < 1, \mathrm{q(z)} = \frac{1+\mathrm{Az}}{1+\mathrm{Bz}}$, in theorem (2) and theorem (3).

Corollary (5):

Let $-1 \leq B \leq A < 1, \partial, \varphi \in \mathbb{C} \setminus \{0\}, Re\frac{1-Az}{1+Bz} > \max\left\{0, -Re\left(\frac{\varphi}{\partial}\right)\right\},$

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if $f \in \mathcal{A}$ will satisfy the condition of subordination:

defined by (18), then $\frac{1+Az}{1+Bz}$ will be the best dominant$\Theta(z) < \frac{1+Az}{1+Bz} + \frac{\partial}{\varphi}\frac{(A-B)z}{(1+Bz)^2}$, $where\ \Theta(z)$

$$(\frac{\gamma_1 I^m(\lambda_1,\lambda_2,l,n)f(z) + \gamma_2 I^m(\lambda_1,\lambda_2,l,n)f(z)}{(\gamma_1+\gamma_2)z})^\alpha \prec \frac{1+Az}{1+Bz}.$$

Corollary (6):

by setting $B = -1, and\ A = 1\ in\ corollary\ (5)$, suppose that

$$Re\frac{1-z}{1+z} > \max\left\{0, -Re\left(\frac{\varphi}{\partial}\right)\right\}, \partial, \varphi \in \backslash \{0\},$$

If $f(z)$ satisfies the following subordination condition $[\Theta(z) \prec \frac{1+z}{1+z} + \frac{\partial}{\varphi}\frac{2z}{(1+z)^2}]$, such that $\Theta(z) is$ obtain by (17), then $\frac{1+z}{1-z}$ will be best dominant and

$$(\frac{\gamma_1 I^m(\lambda_1,\lambda_2,l,n)f(z) + \gamma_2 I^m(\lambda_1,\lambda_2,l,n)f(z)}{(\gamma_1+\gamma_2)z})^\alpha \prec \frac{1+z}{1+z}.$$

4. Superordination Results:

Theorem (4):

Let q(z) be a convex and univalent function in the unit disk $\mathbb{U}$, with

$$0 \neq (\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z}) \in \mathcal{A}[q(0),1] \cap Q$$

and $q(z) \neq 0, q(0) = 1$ for all $z \in \mathbb{U}\ with\ Re\ (\partial) > 0$.

Then the univalent function $\psi(z) = \left(1 - \frac{\vartheta}{x}\right)\left(\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z}\right) + \frac{\vartheta}{x}\left(\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z}\right)$

in $\mathbb{U}$ will be the best subordination of the following equation

$$(22), q(z) + \vartheta z q'(z) \prec \psi(z)$$

and

$$q(z) \prec \left(\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z}\right).$$

Proof:

According to our assumptions, we let $p(z) = \left(\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z}\right), z \in \mathbb{U}$ be analytic in $\mathbb{U}$.

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By taking the differentiation logarithmically with respect to z , so that one obtains

$$\frac{zp'(z)}{p(z)} = \frac{z(I^m(\lambda_1,\lambda_2,l,n)f(z))'}{I^m(\lambda_1,\lambda_2,l,n)f(z)} - 1.$$

By some calculations, we obtain $\psi(z) = p(z) + \vartheta zp'(z), where\ \psi(z)$ is known in (9), and from lemma (3), we get the required result.

Theorem (5):

Assume q (z) be a convex function in $\mathbb{U}$, with $q(z) \neq 0, q(0) = 1$ for all $z \in \mathbb{U}, \alpha, \partial \in \mathbb{C}^*, \lambda_1, \gamma_2, \varphi, \tau \in \mathbb{C}, with\ Re\left(\frac{\varphi}{\vartheta}\right) > 0, and\ \gamma_1 + \gamma_2 \neq 0.$

Let

$$0 \neq \left(\frac{\gamma_1 I^m(\lambda_1,\lambda_2,l,n)f(z) + \gamma_2 I^m(\lambda_1,\lambda_2,l,n)f(z)}{(\gamma_1+\gamma_2)z}\right)^\alpha \in \mathcal{A}[q(0),1] \cap Q.$$

If the univalent q(z) in $\mathbb{U}$ and the known function $\Theta(z)$ in (3.11) satisfy

$$\varphi \qquad (23) q(z) + \partial zq'(z) + \tau \prec \Theta(z),$$

then q(z) will be the best subordinate and

$$q(z) \prec \left(\frac{\gamma_1 I^m(\lambda_1,\lambda_2,l,n)f(z) + \gamma_2 I^m(\lambda_1,\lambda_2,l,n)f(z)}{(\gamma_1+\gamma_2)z}\right)^\alpha.$$

Proof:

Suppose that

$$(24) \xi(z) = \left(\frac{\gamma_1 I^m(\lambda_1,\lambda_2,l,n)f(z)+\gamma_2 I^m(\lambda_1,\lambda_2,l,n)f(z)}{(\gamma_1+\gamma_2)z}\right)^\alpha.$$

By taking the differentiation of the (24) with respect to logarithmically of z, we obtain

$$(25) \frac{z\xi'(z)}{\xi(z)} = \alpha\left(\frac{\gamma_1 z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)' + \gamma_2 z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{\gamma_1 I^m(\lambda_1,\lambda_2,l,n)f(z)+\gamma_2 I^m(\lambda_1,\lambda_2,l,n)f(z)}\right).$$

A simple computation and using (25), we have

$$\left(\frac{\gamma_1 I^m(\lambda_1,\lambda_2,l,n)f(z)+\gamma_2 I^m(\lambda_1,\lambda_2,l,n)f(z)}{(\gamma_1+\gamma_2)z}\right)^\alpha \times [\varphi + \alpha\partial\left(\frac{\gamma_1 z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)' + \gamma_2 z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{\gamma_1 I^m(\lambda_1,\lambda_2,l,n)f(z)+\gamma_2 I^m(\lambda_1,\lambda_2,l,n)f(z)}\right) - 1)] + \tau = \varphi\ \xi(z) + \partial z\xi'(z) + \tau.$$

For lemma (2.4), we have the required result.

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The next corollary can be get by setting $-1 \leq B \leq A < 1$, $q(z) = \frac{1+Az}{1+Bz}$, in theorem (23).

Corollary (7):

Assume that $(\frac{\gamma_1 I^m(\lambda_1,\lambda_2,l,n)f(z)+\gamma_2 I^m(\lambda_1,\lambda_2,l,n)f(z)}{(\gamma_1+\gamma_2)z})^\alpha \in \mathcal{A}[q(0),1] \cap Q - 1 \leq B \leq A < 1$ and $\mathrm{Re}(\frac{\varphi}{\partial}) > 0$.

If $f(z) \in \mathcal{A}$ holds and under superordination condition with $\Theta(z)$ is univalent function defined by (19)

$$\Theta(z), \frac{1+Az}{1+Bz} + \frac{\partial}{\varphi}\frac{(A-B)z}{(1+Bz)^2} \prec$$

then $\frac{1+Az}{1+Bz} \prec (\frac{\gamma_1 I^m(\lambda_1,\lambda_2,l,n)f(z)+\gamma_2 I^m(\lambda_1,\lambda_2,l,n)f(z)}{(\gamma_1+\gamma_2)z})^\propto$

and $\frac{1+Az}{1+Bz}$ will be the best subordinant.

5. Sandwich Result:

It is important to point out that we can obtain the following final two results, sandwich theorems, by applying the starlike properties in theorem (2) and combining theorem (3) and theorem (4) to consequences of superordination and differential subordination.

Theorem (6):

Assume $q_1(z)$ and $q_2(z)$ are univalent and convex in $\mathbb{U}$ with $1 = q_1(0) \neq q_2(0)$ where $q_1(z)$ and $q_2(z)$ are not equal to zero, $\partial \in \mathbb{C}^*, for\ all\ z \in \mathbb{U}\ and\ 0 \neq \left(\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z}\right) \in \mathcal{A}[1,1] \cap Q$.

Suppose that $\psi(z)$ be univalent function in $\mathbb{U}$, where $\psi(z)$ is given by (3.2) satisfies $q_1(z) + \partial z q'_1(z) \prec \psi(z) \prec q_2(z) + \partial z q'_2(z)$, then $q_1(z) \prec \left(\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z}\right) \prec q_2(z)$, and $q_1(z), q_2(z)$ are to be the best dominant and best subordinant, respectively.

In order to get the next theorem we have to join the result in theorem (3) and theorem (4).

Theorem (7):

Assume $q_1(z) and\ q_2(z)$ are univalent and convex in $\mathbb{U}$ with $1 = q_1(0) = q_2(0)$ where $\left(\frac{\gamma_1 I^m(\lambda_1,\lambda_2,l,n)f(z)+\gamma_2 I^m(\lambda_1,\lambda_2,l,n)f(z)}{(\gamma_1+\gamma_2)z}\right)^\alpha \neq 0$.

And $q_1(z)$, $q_2(z)$ are not equal to zero, $\partial \in \mathbb{C}^*$ for all $z \in \mathbb{U}$ and let $\Theta(z)$ be univalent function in $\mathbb{U}$ that satisfies

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$$\varphi q_1(z) + \partial z {q'}_1(z) + \tau \prec \Theta(z) \prec \varphi q_2(z) + \partial z {q'}_2(z) \,,$$

Then $q_1(z), q_2(z)$ are to be the best dominant and best subordinant, respectively and

$$q_1(z) \prec \left(\frac{\gamma_1 I^m(\lambda_1,\lambda_2,l,n)f(z) + \gamma_2 I^m(\lambda_1,\lambda_2,l,n)f(z)}{(\gamma_1+\gamma_2)z}\right)^{\alpha} \prec q_2(z).$$

These references [1,9,5,6,14,10,12] contain additional research and studies on analytic functions connected to the derivative operator and integral operator. It is advisable to consult these sources for further information and insights.

Conclusion

In this study, we aimed to present original findings about a generalized derivative operator

for a certain category of analytic functions on the open unit disk. Our approach involved the utilization of differential subordination and superordination. The derivation of the theorems and corollaries involved an analysis of relevant lemmas pertaining to differential subordination. The paper revealed unique findings on differential subordination and superordination through the utilization of sandwich theorems.

Also, with the aid of q-calculus, we investigate the concept outlined in this article can be employed to easily study a large range of analytic and univalent functions linked to several theorem. This may open numerous new lines of inquiry into the geometric function theory of complex analysis and appropriate areas.

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