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Analysis For the General Term of a Cauchy Product of Two Series of the Truncation Error of Restrictive Approximations

*Nadia Ismail El bergali [1]
[1]General Department، Faculty of Information Technology، Misurata University .

A B S T R A C T

This paper contains an advanced analytical proof that the zero-truncation error is in the general term of the Cauchy product of two series of the truncation error for some restrictive approximations for IBVP for parabolic and hyperbolic equations. In this paper, it had been proved that the general term of the power series of space and time lengths in the differential form of the truncation error is exactly zero.

Keywords:

Cauchy product, Hyperbolic Equations, Parabolic Equations, Restrictive Taylor, Zero Truncation.

تحليل المصطلح العام لمنتج كوشي لسلسلتين لخطأ الاقتطاع للتقديرات التقييدية

*نادية إسماعيل البرقلي [1]
[1]القسم العام – كلية تقنية المعلومات – جامعة مصراتة.

الملخص

تحتوي هذه الورقة على دليل تحليلي متقدم على أن خطأ الاقتطاع الصفري هو في المصطلح العام لحاصل ضرب كوشي لسلسلتين من خطأ الاقتطاع لبعض التقريبات المقيدة لـ IBVP للمعادلات المكافئة والقطع الزائد. في هذه الورقة، تم إثبات أن المصطلح العام لسلسلة القوة لأطوال المكان والزمان في الشكل التفاضلي لخطأ الاقتطاع هو صفر تمامًا.

Introduction

Very recent two papers [[1]] and [[2]] discussed the convergence of the restrictive Pade' and Restrictive Taylor methods for solution of IBVP for parabolic and hyperbolic types, these papers consider only the first few terms of the local truncation error series.

*Corresponding author:
E-mail addresses nadaismail3301169@gmail.com.Received 13 February 2023 - Received in revised form 2may 2023 - Accepted 6may 2023

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In numerical analysis and scientific computing, truncation error is an error caused by approximating a mathematical process. A truncation error is the difference between an actual and a truncated, or cut-off, value. A truncated quantity is represented by a numeral with a fixed number of allowed digits, with any excess digits chopped off -- hence, the expression truncated.

1. Local Truncation Error (LTE) for the Classical Method for Parabolic Type: Consider the equation:

$$u_t = u_{xx} \; ; \; u(x, 0) = f(x), \; u(0, t) = u(1, t) = 0 \qquad (1)$$

The Crank-Nicolson finite difference is:

$$\frac{1}{k}\left(u_{ij+1}-u_{ij}\right)=\frac{1}{2h^2}\left[\left(u_{i-1}-2u_i+u_{i+1}\right)_j+\left(u_{i-1}-2u_i+u_{i+1}\right)_{j+1}\right]$$

Its LTE finite difference form is:

$$T_{ij/CN}=\frac{1}{k}\left(u_{ij+1}-u_{ij}\right)-\frac{1}{2h^2}\left[\left(u_{i-1}-2u_i+u_{i+1}\right)_j\right.$$

Using Taylor expansion, with ascending powers of h and k the LTE finite difference form becomes:

$$T_{ij/CN}=\frac{1}{k}\left[u_{ij}+ku_t+\frac{k^2}{2!}u_{t^2}+\frac{k^3}{3!}u_{t^2}+\frac{k^4}{4!}u_{t^2}+\frac{k5}{5!}u_{t^2}+\frac{k^6}{6!}u_{t^2}+\frac{k^7}{7!}u_{t^2}+\ldots-u_{ij}\right]$$

$$-\frac{1}{2h^2}\left\{u_{ij}-hu_x+\frac{h^2}{2!}u_{x^2}-\frac{h^3}{3!}u_{x^3}+\frac{h^4}{4!}u_{x^4}-\frac{h^5}{5!}u_{x^5}+\frac{h^6}{6!}u_{x^6}-\frac{h^7}{7!}u_{x^7}\right.$$

$$-2u_{ij}\quad +u_{ij}+hu_x+\frac{h^2}{2!}u_{x^2}+\frac{h^3}{3!}u_{x^3}+\frac{h^4}{4!}u_{x^4}+\frac{h^5}{5!}u_{x^5}+\frac{h^6}{6!}u_{x^6}+\frac{h^7}{7!}u_{x^7}+\ldots$$

$$+u_{ij}+ku_t-hu_x+\frac{1}{2!}\left[k^2u_{t^2}-2hku_{xt}+k^2u_{x^2}\right]$$

$$+\frac{1}{3!}\left[k^3u_{t^3}-3k^2hu_{t^2x}+3kh^2u_{tx^2}-h^3u_{x^3}\right]$$

$$+\frac{1}{4!}\left[k^4u_{t^4}-4k^3hu_{t^3x}+6k^2h^2u_{t^2x^2}-4kh^3u_{tx^3}+h^4u_{x^4}\right]$$

$$+\frac{1}{5!}\left[k^5u_{t^5}-5k^4hu_{t^4x}+10k^3h^2u_{t^3x^2}-10k^2h^3u_{t^2x^3}+5kh^4u_{tx^4}-h^5u_{x^5}\right]$$

$$+\frac{1}{6!}\left[k^6u_{t^6}-6k^5h\,u_{t^5x}+15k^4h^2u_{t^4x^2}-20k^3h^3u_{t^3x^3}\right.$$

$$\left.+15k^2h^4u_{t^2x^4}-6kh^5u_{tx^5}+h^6u_{x^6}\right]$$

$$+\frac{1}{7!}\left[k^7u_{t^7}-7k^6h\,u_{t^6x}+21k^5h^2u_{t^5x^2}-35k^4h^3u_{t^4x^3}\right.$$

$$\left.+35k^3h^4u_{t^3x^4}-21k^2h^5u_{t^2x^5}+7kh^6u_{tx^6}-h^7u_{x^7}\right]+\ldots$$

$$-2\left[u_{ij}+ku_t+\frac{k^2}{2!}u_{t^2}+\frac{k^3}{3!}u_{t^2}+\frac{k^4}{4!}u_{t^2}+\frac{k^5}{5!}u_{t^2}+\frac{k^6}{6!}u_{t^2}+\frac{k^7}{7!}u_{t^2}+\ldots\right]$$

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$$
\begin{aligned}
&+u_{ij}+ku_t+hu_x+\frac{1}{2!}\left[k^2u_{t^2}+2hku_{xt}+k^2u_{x^2}\right]\\
&+\frac{1}{3!}\left[k^3u_{t^3}+3k^2hu_{t^2x}+3kh^2u_{tx^2}+h^3u_{x^3}\right]\\
&+\frac{1}{4!}\left[k^4u_{t^4}+4k^3hu_{t^3x}+6k^2h^2u_{t^2x^2}+4kh^3u_{tx^3}+h^4u_{x^4}\right]\\
&+\frac{1}{5!}\left[k^5u_{t^5}+5k^4hu_{t^4x}+10k^3h^2u_{t^3x^2}+10k^2h^3u_{t^2x^3}+5kh^4u_{tx^4}+h^5u_{x^5}\right]\\
&+\frac{1}{6!}\left[k^6u_{t^6}+6k^5h\,u_{t^5x}+15k^4h^2u_{t^4x^2}+20k^3h^3u_{t^3x^3}\right.\\
&\qquad\left.+15k^2h^4u_{t^2x^4}+6kh^5u_{tx^5}+h^6u_{x^6}\right]\\
&+\frac{1}{7!}\left[k^7u_{t^7}+7k^6h\,u_{t^6x}+21k^5h^2u_{t^5x^2}+35k^4h^3u_{t^4x^3}\right.\\
&\left.+35k^3h^4u_{t^3x^4}+21k^2h^5u_{t^2x^5}+7kh^6u_{tx^6}+h^7u_{x^7}\right]+\ldots\Big\}
\end{aligned}
$$

Then we get:

$$
\begin{aligned}
T_{ij/CN}=&\frac{k}{2!}u_{t^2}+\frac{k^2}{3!}u_{t^2}+\frac{k^3}{4!}u_{t^2}+\frac{k^4}{5!}u_{t^2}+\frac{k^5}{6!}u_{t^2}+\ldots\\
&-2\left[\frac{h^2}{4!}u_{x^4}+\frac{h^4}{6!}u_{x^6}+\frac{h^6}{8!}u_{x^8}+\ldots\right]\\
&-\frac{k}{2}u_{tx^2}-\frac{k^2}{4}u_{t^2x^2}-\frac{k^3}{12}u_{t^3x^2}-\frac{k^4}{48}u_{t^4x^2}\\
&-\frac{{}^7C_2k^5}{7!}u_{t^5x^2}-\frac{{}^8C_2k^6}{8!}u_{t^6x^2}-\frac{{}^9C_2k^7}{9!}u_{t^7x^2}-\ldots\\
&-\frac{kh^2}{24}u_{tx^4}-\frac{k^2h^2}{48}u_{t^2x^4}-\frac{35k^3h^2}{7!}u_{t^3x^4}-\frac{{}^8C_4k^4h^2}{8!}u_{t^4x^4}\\
&-\frac{{}^9C_4k^5h^2}{9!}u_{t^5x^4}-\frac{{}^{10}C_4k^6h^2}{10!}u_{t^6x^4}-\ldots\\
&-\frac{{}^7C_6\,kh^4}{7!}u_{tx^6}-\frac{{}^8C_6k^2h^4}{8!}u_{t^2x^6}-\frac{{}^9C_6k^3h^4}{9!}u_{t^3x^6}\\
&-\frac{{}^{10}C_6k^4h^4}{10!}u_{t^4x^6}-\frac{{}^{11}C_6k^5h^4}{11!}u_{t^5x^6}-\ldots
\end{aligned}
$$

Accordingly, we get:

$$
T_{ij/CN}=\frac{k}{2\ !}u_{t^2}+\frac{k^2}{3\ !}u_{t^3}+\frac{k^3}{4\ !}u_{t^4}+\frac{k^4}{5\ !}u_{t^5}+\frac{k^5}{6\ !}u_{t^6}+\ldots
$$

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$$-\frac{2h^2}{4\ !}u_{x^4}-\frac{2h^4}{6\ !}u_{x^6}-\frac{2h^6}{8\ !}u_{x^8}-\frac{2h^8}{10\ !}u_{x^{10}}-\dots$$

$$-\frac{{}^3C_2k}{3\ !}u_{t\,x^2}-\frac{{}^4C_2k^2}{4\ !}u_{t^2x^2}-\frac{{}^5C_2k^3}{5\ !}u_{t^3x^2}-\frac{{}^6C_2k^4}{6\ !}u_{t^4x^2}-\dots$$

$$-\frac{{}^5C_4kh^2}{5\ !}u_{t\,x^4}-\frac{{}^6C_4k^2h^2}{6\ !}u_{t^2x^4}-\frac{{}^7C_4k^3h^2}{7\ !}u_{t^3x^4}-\frac{{}^8C_4k^4h^2}{8\ !}u_{t^4x^4}-\dots$$

$$-\frac{{}^7C_6kh^4}{7\ !}u_{t\,x^6}-\frac{{}^8C_6k^2h^4}{8\ !}u_{t^2x^6}-\frac{{}^9C_6k^3h^4}{9\ !}u_{t^3x^6}-\frac{{}^{10}C_6k^4h^4}{10\ !}u_{t^4x^6}-\dots$$

-…. .

$$=\sum_{n=2}^{\infty}\frac{k^{n-1}}{n\ !}u_{t^n}-2\sum_{n=2}^{\infty}\frac{h^{2(n-1)}}{(2n)!}u_{x^{2n}}-\sum_{n=1}^{\infty}\frac{{}^{n+2}C_2k^n}{(n+2)!}u_{x^2t^n}$$

$$-\sum_{n=1}^{\infty}\frac{{}^{n+4}C_4k^nh^2}{(n+4)!}u_{x^4t^n}-\sum_{n=1}^{\infty}\frac{{}^{n+6}C_6k^nh^4}{(n+6)!}u_{x^6t^n}-\dots$$

.

$$=[\sum_{2}^{\infty}\frac{k^{n-1}}{n\ !}u_{t^n}-2\sum_{m=2}^{\infty}\frac{h^{2(m-1)}}{(2m)\ !}u_{x^{2m}}-\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{{}^{n+2m}C_{2m}k^nh^{2(m-1)}}{(n+2m)\ !}u_{x^{2m}t^n}]_{ij}$$

$$=\left[\sum_{n=2}^{\infty}\frac{k^{n-1}}{n!}u_{t^n}-2\sum_{m=2}^{\infty}\frac{h^{2(m-1)}}{(2m)!}u_{x^{2m}}-\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{k^nh^{2(m-1)}}{n!(2m)!}u_{x^{2m}t^n}\right]_{ij}\quad(3)$$

2. The LTE for The Restrictive Pade` Approximation for IBVP for Parabolic Type: [3]

By computing the jth and zeroth levels for Crank-Nicolson method, its truncation errors are $T_{ij/CN}$ and $T_{i0/CN}$. There finite difference forms are:[4]

$$T_{ij/CN}=\frac{1}{k}\left[\left(-\frac{1}{2}ru_{i-1}+(1+r)u_i-\frac{1}{2}ru_{i+1}\right)_{j+1}-\left(\frac{1}{2}ru_{i-1}+(1-r)u_i+\frac{1}{2}ru_{i+1}\right)_j\right]\quad(4)$$

Similarly,

$$T_{i0/CN}=\frac{1}{k}\left[\left(-\frac{1}{2}ru_{i-1}+(1+r)u_i-\frac{1}{2}ru_{i+1}\right)_1-\left(\frac{1}{2}ru_{i-1}+(1-r)u_i+\frac{1}{2}ru_{i+1}\right)_0\right]$$

By computing the jth and zeroth levels for restrictive Pade` method , its truncation errors are $T_{ij/RP}$ and $T_{i0/RP}$ .There finite difference forms are:

$$T_{ij/RP}=\frac{1}{k}\left((\varepsilon_i-\frac{1}{2})\ r\ \ u_{i-1}+\left(1+(1+\varepsilon_i)r\right)u_i+(\varepsilon_i-\frac{1}{2})\ r\ \ u_{i+1}\right)_{j+1}$$

$$-\frac{1}{k}\left((\varepsilon_i+\frac{1}{2})\ r\ \ u_{i-1}+\left(1+(\varepsilon_i-1)r\right)u_i+(\varepsilon_i+\frac{1}{2})\ r\ \ u_{i+1}\right)_j\quad(5)$$

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Then we get:

$$T_{ij/RP} = \frac{r\varepsilon_i}{k}\sum_{l=i-1}^{i+1}\left(u_l^{j+1} - u_l^{j}\right) + T_{ij/CN}$$

i.e. $T_{ij/RP} = r\ å_i ó_{ij} + T_{ij/CN}$ (6)

where:

$$= \frac{1}{k}\left[u_{i-1}^{j+1} - u_{i-1}^{j} + u_i^{j+1} - u_i^{j} + u_{i+1}^{j+1} - u_{i+1}^{j}\right]$$

$$\sigma_{ij} = \frac{1}{k}\sum_{l=i-1}^{i+1}\left(u_l^{j+1} - u_l^{j}\right)$$

. (7)

Using Taylor expansion and grouping them with ascending powers of h and k:

$$\sigma_{ij} = \left(3u_t + h^2 u_{tx^2} + 2\frac{h^4}{4\ !}u_{tx^4} + 2\frac{h^6}{6\ !}u_{tx^6} + \ldots\right)$$

$$+ \frac{1}{2!}k\left(3u_{t^2} + h^2 u_{t^2x^2} + 2\frac{h^4}{4\ !}u_{t^2x^4} + 2\frac{h^6}{6\ !}u_{t^2x^6} + \ldots\right)$$

$$+ \frac{1}{3\ !}k^2\left(3u_{t^3} + h^2 u_{t^3x^2} + 2\frac{h^4}{4\ !}u_{t^3x^4} + 2\frac{h^6}{6\ !}u_{t^3x^6} + \ldots\right)$$

$$+ \frac{1}{4\ !}k^3\left(3u_{t^4} + h^2 u_{t^4x^2} + 2\frac{h^4}{4\ !}u_{t^4x^4} + 2\frac{h^6}{6\ !}u_{t^4x^6} + \ldots\right) + \ldots .$$

Or, in the series summation form:

$$\sigma_{ij} = \sum_{n=1}^{\infty}\frac{k^{n-1}}{n\ !}\left(3u_{t^n} + 2\sum_{m=1}^{\infty}\frac{h^{2m}}{(2m)\ !}u_{t^n x^{2m}}\right)_{ij} . \qquad (8)$$

Similarly,

$T_{i0/RP} = r\ å_i ó_{i0} + T_{i0/CN}$ (9)

where:

$$\sigma_{i0} = \frac{1}{k}\sum_{l=i-1}^{i+1}\left(u_l^{1} - u_l^{0}\right)$$

, or

$$= \sum_{n=1}^{\infty}\frac{k^{n-1}}{n\ !}\left(3u_{t^n} + 2\sum_{m=1}^{\infty}\frac{h^{2m}}{(2m)\ !}u_{t^n x^{2m}}\right)_{i0}$$

The main idea of the restrictive Pade` method is to put $T_{i0/RP} = 0$ to get $å_i$, i.e. equation (9) will take the form:

$T_{i0/CN} = -\ r\ å_i ó_{i0}$. (10)

Eliminating $å_i$ from equation (6) and (10) we get

$ó_{i0}\ T_{ij/RP} = T_{ij/CN}\ ó_{i0} - T_{i0/CN}\ ó_{ij}$ (11)

Avoiding the complexity of this equation by using the Cauchy product of two series (B.4), we get Crank-Nicolson truncation error series relation as follows:

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$$
T_{ij/CN}\,\delta_{i0} = \Big[\sum_{2}^{\infty}\frac{k^{n-1}}{n\,!}u_{t^n} - 2\sum_{m=2}^{\infty}\frac{h^{2(m-1)}}{(2m)!}u_{x^{2m}} - \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{k^n h^{2(m-1)}}{n!(2m)!}u_{x^{2m}t^n}\Big]_{ij}
$$
$$
\times\Big[\sum_{n=1}^{\infty}\frac{k^{n-1}}{n\,!}\left(3u_{t^n} + 2\sum_{m=1}^{\infty}\frac{h^{2m}}{(2m)!}u_{t^n x^{2m}}\right)\Big]_{i0}
$$
$$
= 3\sum_{n=1}^{\infty}\sum_{r=1}^{n}\frac{k^r k^{n-r-1}}{(r+1)\,!(n-r)\,!}\left(u_{t^{r+1}}\right)_{ij}\left(u_{t^{n-r}}\right)_{io}
$$
$$
-6\sum_{n=1}^{\infty}\sum_{r=1}^{n}\frac{h^{2r}k^{n-r-1}}{(2r+2)\,!(n-r)\,!}\left(u_{x^{2r+2}}\right)_{ij}\left(u_{t^{n-r}}\right)_{io}
$$
$$
-3\sum_{n=1}^{\infty}\sum_{r=1}^{n}\sum_{m=1}^{\infty}\frac{k^r k^{n-r-1}h^{2(m-1)}}{r\,!(n-r)\,!(2m)\,!}\left(u_{x^{2m}t^r}\right)_{ij}\left(u_{t^{n-r}}\right)_{io}
$$
$$
+2\sum_{n=1}^{\infty}\sum_{r=1}^{n}\sum_{m=1}^{\infty}\frac{k^r k^{n-r-1}h^{2m}}{(r+1)!(n-r)\,!(2m)\,!}\left(u_{t^{n-r}x^{2m}}\right)_{i0}\left(u_{t^{r+1}}\right)_{ij}
$$
$$
-4\sum_{n=1}^{\infty}\sum_{r=1}^{n}\sum_{m=1}^{\infty}\frac{h^{2r}k^{n-r-1}h^{2m}}{(2r+2)\,!(n-r)\,!(2m)\,!}\left(u_{t^{n-r}x^{2m}}\right)_{i0}\left(u_{x^{2r+2}}\right)_{ij}
$$
$$
-2\sum_{n=1}^{\infty}\sum_{r=1}^{n}\sum_{p=1}^{\infty}\sum_{m=1}^{\infty}\frac{h^{2(p-1)}k^r k^{n-r-1}h^{2m}}{r\,!(2p)\,!(n-r)\,!(2m)\,!}\left(u_{t^{n-r}x^{2m}}\right)_{i0}\left(u_{x^{2p}t^r}\right)_{ij}.
$$

Similarly, the form for $T_{i0/CN}\delta_{ij}$.

Substituting in (7) we reduce the restrictive Pade` truncation error in the following relation:

$$
\delta_{i0}\,T_{ij/RP} = 3\sum_{n=1}^{\infty}\sum_{r=1}^{n}\frac{k^{n-1}}{(r+1)\,!(n-r)\,!}\left[\left(u_{t^{r+1}}\right)_{ij}\left(u_{t^{n-r}}\right)_{io} - \left(u_{t^{r+1}}\right)_{i0}\left(u_{t^{n-r}}\right)_{ij}\right]
$$
$$
-6\sum_{n=1}^{\infty}\sum_{r=1}^{n}\frac{h^{2r}k^{n-r-1}}{(2r+2)\,!(n-r)\,!}\left[\left(u_{x^{2r+2}}\right)_{ij}\left(u_{t^{n-r}}\right)_{io} - \left(u_{x^{2r+2}}\right)_{i0}\left(u_{t^{n-r}}\right)_{ij}\right]
$$
$$
-3\sum_{n=1}^{\infty}\sum_{r=1}^{n}\sum_{m=1}^{\infty}\frac{k^{n-1}h^{2(m-1)}}{r\,!(n-r)\,!(2m)\,!}\left[\left(u_{x^{2m}t^r}\right)_{ij}\left(u_{t^{n-r}}\right)_{io} - \left(u_{x^{2m}t^r}\right)_{i0}\left(u_{t^{n-r}}\right)_{ij}\right]
$$
$$
+2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{r=1}^{n}\frac{k^{n-1}h^{2m}}{(r+1)\,!(n-r)\,!(2m)\,!}\left[\left(u_{t^{r+1}}\right)_{ij}\left(u_{t^{n-r}x^{2m}}\right)_{i0} - \left(u_{t^{r+1}}\right)_{i0}\left(u_{t^{n-r}x^{2m}}\right)_{ij}\right]
$$
$$
-4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{r=1}^{n}\frac{k^{n-r-1}h^{2m+2r}}{(2r+2)\,!(n-r)\,!(2m)\,!}\left[\left(u_{x^{2r+2}}\right)_{ij}\left(u_{t^{n-r}x^{2m}}\right)_{i0} - \left(u_{x^{2r+2}}\right)_{i0}\left(u_{t^{n-r}x^{2m}}\right)_{ij}\right]
$$
$$
-2\sum_{p=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{r=1}^{n}\frac{k^{n-1}h^{2(p+m-1)}}{r!(2p)!(n-r)!(2m)!}\left[\left(u_{x^{2p}t^r}\right)_{ij}\left(u_{t^{n-r}x^{2m}}\right)_{i0} - \left(u_{x^{2p}t^r}\right)_{i0}\left(u_{t^{n-r}x^{2m}}\right)_{ij}\right]. \quad (12)
$$

Using the relation $\frac{\partial^2}{\partial x^2} = \frac{\partial}{\partial t}$ we can reduce (12) to the following form:

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ói0 $T_{ij/RP}$=
$$3\sum_{n=1}^{\infty}\sum_{r=1}^{n}\frac{k^{n-1}}{(r+1)\ !(n-r)\ !}\left[\left(u_{t^{r+1}}\right)_{ij}\left(u_{t^{n-r}}\right)_{io}-\left(u_{t^{r+1}}\right)_{i0}\left(u_{t^{n-r}}\right)_{ij}\right]$$
$$-6\sum_{n=1}^{\infty}\sum_{r=1}^{n}\frac{k^{n-r-1}h^{2r}}{(2r+2)\ \ !(n-r)\ !}\left[\left(u_{t^{r+1}}\right)_{ij}\left(u_{t^{n-r}}\right)_{i0}-\left(u_{t^{r+1}}\right)_{i0}\left(u_{t^{n-r}}\right)_{ij}\right]$$
$$-3\sum_{n=1}^{\infty}\sum_{r=1}^{n}\sum_{m=1}^{\infty}\frac{k^{n-1}h^{2(m-1)}}{r\ \ !(n-r)\ \ !(2m)\ !}\left[\left(u_{t^{r+m}}\right)_{ij}\left(u_{t^{n-r}}\right)_{i0}-\left(u_{t^{r+m}}\right)_{i0}\left(u_{t^{n-r}}\right)_{ij}\right]$$
$$+2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{r=1}^{n}\frac{k^{n-1}h^{2m}}{(r+1)\ \ !(n-r)\ \ !(2m)\ !}\left[\left(u_{t^{r+1}}\right)_{ij}\left(u_{t^{n-r+m}}\right)_{i0}-\left(u_{t^{r+1}}\right)_{i0}\left(u_{t^{n-r+m}}\right)_{ij}\right]$$
$$-4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{r=1}^{n}\frac{k^{n-r-1}h^{2m+2r}}{(2r+2)\ \ !(n-r)\ \ !(2m)\ !}\left[\left(u_{t^{r+1}}\right)_{ij}\left(u_{t^{n-r+m}}\right)_{i0}-\left(u_{t^{r+1}}\right)_{i0}\left(u_{t^{n-r+m}}\right)_{ij}\right]$$

$$-2\sum_{p=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{r=1}^{n}\frac{k^{n-1}h^{2(p+m-1)}}{r\ \ !(2p)\ \ !(n-r)\ \ !(2m)\ !}\left[\left(u_{t^{p+r}}\right)_{ij}\left(u_{t^{n-r+m}}\right)_{i0}-\left(u_{t^{p+r}}\right)_{i0}\left(u_{t^{n-r+m}}\right)_{ij}\right]$$
(13)

However, we can express (3) and (8) in the following more simple and general forms:

$$T_{ij/CN}=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\alpha_{mn}k^{n-\delta_{0m}}h^{2m-2}\left(u_{x^{2m}t^{n}}\right)_{ij}, \qquad (14)$$

where,

á00 = á01 = á10 = 0,

$$\alpha_{m0}=\frac{-2}{(2m)\ !},\quad \alpha_{0n}=\frac{h^2}{n\ \ !}\quad \forall\ m, n = 2(1)\infty,$$

and $\alpha_{mn}=\dfrac{-1}{n\ \ !\ \ (2m)\ !}$ $\forall$ m, n = 1(1)∞ ,

and äij denotes the Kroniker delta i.e $\delta_{ij}=\begin{cases}1 & i=j\\ 0 & i\neq j\end{cases}$,

$$\sigma_{ij}=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\beta_{mn}k^{n-1}h^{2m}\left(u_{x^{2m}t^{n}}\right)_{ij} \qquad (15)$$

Where:

âm0 = 0 ∀m = 0(1)∞, $\beta_{0n}=\dfrac{3}{n\ \ !}$ ∀ n = 1(1)∞,

and: $\beta_{mn}=\dfrac{2}{n\ !\ (2m)\ !}$ ∀ m, n = 1(1)∞.

So, we get:

$$T_{ij/CN}\sigma_{io}=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{r=0}^{m}\alpha_{rn}\beta_{(m-r)p}k^{n+p-1-\delta_{r0}}h^{2m-2}\left(u_{x^{2r}t^{n}}\right)_{ij}\left(u_{x^{2m-2r}t^{p}}\right)_{io}$$

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and:

$$T_{io/CN}\sigma_{ij} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{r=0}^{m}\alpha_{rn}\beta_{(m-r)p}k^{n+p-1-\delta_{r0}}h^{2m-2}\left(u_{x^{2r}t^{n}}\right)_{io}\left(u_{x^{2m-2r}t^{p}}\right)_{ij}$$

hence (13) will take the form:

$$\sigma_{io}T_{ij/RP} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{r=0}^{m}\alpha_{rn}\beta_{(m-r)p}k^{n+p-1-\delta_{r0}}h^{2m-2}\times \left[\left(u_{x^{2r}t^{n}}\right)_{ij}\left(u_{x^{2m-2r}t^{p}}\right)_{io}-\left(u_{x^{2r}t^{n}}\right)_{io}\left(u_{x^{2m-2r}t^{p}}\right)_{ij}\right]. \quad (16)$$

For $\frac{\partial^2}{\partial x^2}=\frac{\partial}{\partial t}$, equation (16) will take the following simple form:

$$\sigma_{io}T_{ij/RP} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{r=0}^{m}\alpha_{rn}\beta_{(m-r)p}k^{n+p-1-\delta_{r0}}h^{2m-2}\times A_{pm}^{nr} \quad (17)$$

where: $$A_{pm}^{nr} = \left[\left(u_{t^{n+r}}\right)_{ij}\left(u_{t^{p+m-r}}\right)_{io}-\left(u_{t^{n+r}}\right)_{io}\left(u_{t^{p+m-r}}\right)_{ij}\right]. \quad (18)$$

Then for parabolic problems of continuous exact solution $u(x,t)=e^{-c^2t}f(x)$, we can easily show that $A_{pm}^{nr}=0$.

3 LTE for The Classical Method for Hyperbolic Type: [5]

Consider the equation: $u_t + au_x = 0$ (19)

The Crank Nicolson finite difference form is

$$\frac{1}{k}\left(u_{ij+1}-u_{ij}\right)+\frac{a}{4h}\left[\left(u_{i+1}-u_{i-1}\right)_j+\left(u_{i+1}-u_{i-1}\right)_{j+1}\right]=0.$$

Its LTE is $T_{ij/CN} = \frac{1}{k}\left(u_{ij+1}-u_{ij}\right)+\frac{a}{4h}\left[\left(u_{i+1}-u_{i-1}\right)_j+\left(u_{i+1}-u_{i-1}\right)_{j+1}\right]$ (20)

Using Taylor expansion and dividing them with ascending powers of h and k then, as we deal with the preceding part, the LTE finite difference form becomes:

$$T_{ij/CN} = \sum_{n=2}^{\infty}\frac{k^{n-1}}{n\ !}u_{t^n}+\frac{a}{h}\sum_{m=2}^{\infty}\frac{h^{2m-1}}{(2m-1)\ !}u_{x^{2m-1}}+\frac{a}{2h}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{k^n h^{2m-1}}{(2m-1)\ !n\ !}u_{x^{2m-1}t^n} \quad (21)$$

which can be expressed in the following more simple and general form:

$$T_{ij/CN} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\alpha_{mn}h^{2m-2}k^{n-\delta_{0m}}u_{x^{2m-1}t^n} \quad (22)$$

where

á00 = á01 = á10 = 0,

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$$\alpha_{m0} = \frac{a}{(2m-1)\ !},\quad \alpha_{0n} = \frac{h^2}{n\ !}\qquad \forall\ m, n = 2(1)\infty,$$

$$\alpha_{mn} = \frac{a}{2(2m-1)\ !\ \ n\ !}\qquad \forall n, m = 1(1)\ \infty,$$

and $u_{x^m t^n} = \frac{\partial^{n+m}}{\partial x^m t^n}$ ∀ m ≥ 1, ∀n ≥ 0 and u_{x^m} = u if m < 1.

For $u_t + a\,u_x = 0$, then $u_{x^n} = \left(\frac{-1}{a}\right)^n u_{t^n}$ ∀ n = 1(1)∞,

hence $T_{ij/CN} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \alpha_{mn} h^{2m-2} k^{n-\delta_{0m}} u_{t^{2m+n-1}}$,

where

$\acute{a}_{00} = \acute{a}_{01} = \acute{a}_{10} = 0$,

$$\alpha_{m0} = \frac{(-1)^{2m-1}}{a^{2m-2}(2m-1)\ !},\quad \alpha_{0n} = \frac{h^2}{n\ !}\qquad \forall\ m, n = 2(1)\infty,$$

and $\alpha_{mn} = \frac{(-1)^{2m-1}}{2\ \ a^{2m-2}\ \ n\ !\ \ (2m-1)\ !}$ ∀ n, m = 1(1) ∞.

4. LTE for Restrictive Pade' Approximation for IBVP of Hyperbolic Type: [6]

By computing the jth and zeroth levels for Crank-Nicolson method , its LTEs are $T_{ij/CN}$ and $T_{i0/CN}$.There finite difference forms for $r = \frac{ka}{2h}$ are:

$$T_{ij/CN} = \frac{1}{2k}\left[\left(-ru_{i-1} + 2u_i + ru_{i+1}\right)_{j+1} - \left(ru_{i-1} + 2u_i - ru_{i+1}\right)_j\right] \tag{24}$$

Similarly,

$$T_{i0/CN} = \frac{1}{2k}\left[\left(-ru_{i-1} + 2u_i + ru_{i+1}\right)_1 - \ \left(ru_{i-1} + 2u_i - ru_{i+1}\right)_0\right]$$

The Corresponding LTE for restrictive Pade' method using $r = \frac{ka}{2h}$, equation (24) will take the form:

$$T_{ij/RP} = \frac{1}{2k}\left[\left(-ru_{i-1} + 2(1+\varepsilon_i\ \ r)u_i + ru_{i+1}\right)_{j+1} - \left(r\ \ u_{i-1} + 2(1+\varepsilon_i\ \ r)\ \ u_i - ru_{i+1}\right)_j\right]$$

$$T_{ij/RP} = \frac{r}{k}\left(u_{ij+1} - u_{ij}\right)\mathring{a}_i + T_{ij/CN}\ . \tag{25}$$

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Then $T_{ij/RP} = \frac{a}{2h}\left(u_{ij+1}-u_{ij}\right)$ å$_i$ + $T_{ij/CN}$
= ó$_{ij}$ å$_i$ + $T_{ij/CN}$, (26)

Where:

$$\text{ó}_{ij} = \frac{a}{2h}\left(u_{ij+1}-u_{ij}\right) = \frac{a}{2h}\sum_{n=1}^{\infty}\frac{k^n}{n!}\left(u_{t^n}\right)_{ij} = \frac{1}{h}\sum_{n=0}^{\infty}\beta_n k^n\left(u_{t^n}\right)_{ij}. \quad (27)$$

for: $\beta_n = \frac{a}{2(n\ \ !)}$ ∀ n > 0 and â$_0$ = 0.

Similarly,

$T_{i0/RP} = T_{i0/CN}$ + ó$_{i0}$ å$_i$. (28)

Applying the restrictive Pade' method to find å$_i$ we p

then we get

ó$_{i0}$ å$_i$ = -$T_{i0/CN}$. (29)

Eliminating å$_i$ from equation (26) and (29) we get

ó$_{i0}$ $T_{ij/RP}$ = $T_{ij/CN}$ ó$_{i0}$ - $T_{i0/CN}$ ó$_{ij}$. (30)

By (22), (27) and (30) using (28) we get

$$\text{ó}_{i0}\, T_{ij/RP} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{n}\alpha_{mr}\beta_{n-r}h^{2m-1}k^{n-\delta_{0m}}\left[\left(u_{t^{2m+r-1}}\right)_{ij}\left(u_{t^{n-r}}\right)_{i0}-\left(u_{t^{2m+r-1}}\right)_{ij}\left(u_{t^{n-r}}\right)_{ij}\right]$$

Putting $A_m^{nr} = \left[\left(u_{t^{2m+r-1}}\right)_{ij}\left(u_{t^{n-r}}\right)_{i0}-\left(u_{t^{2m+r-1}}\right)_{ij}\left(u_{t^{n-r}}\right)_{ij}\right]$.

Then for hyperbolic problems of continuous exact solution $u(x,t)=e^{-c^2t}f(x)$, we can easily get $A_m^{nr}=0$.

5 .LTE for Restrictive Taylor Approximation for Solving IBVP of Parabolic Type: [7]

Consider the equation: $u_t = u_{xx}$ (31)

The finite difference equation is:

$$\frac{1}{k}\left(u_{ij+1}-u_{ij}\right) = \frac{1}{h^2}\left(u_{i-1}-2u_i+u_{i+1}\right)_j$$

Its local truncation error finite difference form is:

$$T_{ij/CN} = \frac{1}{k}\left(u_{ij+1}-u_{ij}\right) - \frac{1}{h^2}\left(u_{i-1}-2u_i+u_{i+1}\right)_j \quad (32)$$

Its local truncation error finite differential form is:

$$T_{ij/CN} = \sum_{n=1}^{\infty}\left[\frac{k^n}{(n+1)\ \ !}-\frac{2\ \ h^{2n}}{(2n+2)\ \ !}\right]u_{t^{n+1}} = \sum_{n=0}^{\infty}\alpha_n u_{t^{n+1}} \quad (33)$$

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where $\acute{a}_n = \frac{k^n}{(n+1)\ !} - \frac{2h^{2n}}{(2n+2)\ !}$ ∀ n = 1 (1)∞, $\acute{a}_0 = 0$, (34)

By computing the j[th] and zero[th] levels for restrictive Taylor method, its truncation errors are $T_{ij/CN}$ and $T_{i0/CN}$.

As we deal with the preceding two cases we get:

$$\sigma_{ij} = \sum_{n=0}^{\infty} \beta_n \left(u_{t^{n+1}}\right)_{ij}, \tag{35}$$

where $\beta_n = \frac{2\quad h^{2n+2}}{(2n+2)\ !}$, ∀ n = 1(1)∞ , $\hat{a}_0 = 0$, (36)

then we get

$$\sigma_{i0} T_{ij/RT} = \sum_{n=0}^{\infty} \sum_{r=0}^{n} \alpha_r \beta_{n-r} \left[\left(u_{t^{r+1}}\right)_{ij} \left(u_{t^{n-r+1}}\right)_{i0} - \left(u_{t^{r+1}}\right)_{i0} \left(\right. \right.$$

Putting $A_{nr} = \left[\left(u_{t^{r+1}}\right)_{ij} \left(u_{t^{n-r+1}}\right)_{i0} - \left(u_{t^{r+1}}\right)_{i0} \left(u_{t^{n-r+1}}\right)_{ij} \right]$,

then for parabolic problems of continuous exact solution $u(x,t) = e^{-c^2 t} f(x)$,

we can show that $A_{nr} = 0$.

4.6 LTE for Restrictive Taylor Approximation for Solving IBVP for Hyperbolic Type:

Consider the equation

$$u_t + a u_x = 0 \tag{38}$$

The classical finite difference equation is

$$\frac{1}{k}\left(u_{ij+1} - u_{ij}\right) = \frac{a}{h}\left(u_{i-1j} - u_{ij}\right).$$

Its local truncation error is

$$T_{ij/C} = \frac{1}{k}\left(u_{ij+1} - u_{ij}\right) - \frac{a}{h}\left(u_{i-1j} - u_{ij}\right). \tag{39}$$

As we dealt with the preceding parts, we get

$$T_{ij/C} = \sum_{n=1}^{\infty} \alpha_n \left(u_{t^{n+1}}\right)_{ij} \tag{40}$$

where:

$$\alpha_n = \frac{k^n}{(n+1)\ !} + \frac{(-1)^{n+1} h^n}{(n+1)\ ! a^n}.$$

The corresponding restrictive Taylor approximation is:

$$\frac{1}{k}\left(u_{ij+1} - u_{ij}\right) = \frac{a\varepsilon_i}{h}\left(u_{i-1j} - u_{ij}\right) \tag{41}$$

or,

$$(u_{i,j+1} - u_{i,j}) = r\varepsilon_{i,l}(u_{i-1,j} - u_{i,j}) \tag{42}$$

By computing the j[th] and zero[th] levels for restrictive Taylor method,

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its truncation errors are $T_{ij/RT}$ and $T_{i0/RT}$ where

$$kT_{ij/R} = (u_{ij+1} - u_{ij}) - r\varepsilon_i(u_{i-1j} - u_{ij}) \quad (43)$$

$$kT_{i0/R} = (u_{i1} - u_{i0}) - r\varepsilon_i(u_{i-10} - u_{i0}) \quad (44)$$

Putting $\sigma_{ij} = (u_{i-1j} - u_{ij})$ we get (45)

$$\sigma_{ij} = \sum_{n=1}^{\infty} \beta_n \left(u_{t^n}\right)_{ij}, \quad (46)$$

where:

$$\beta_n = \frac{1}{n!}\left(\frac{h}{a}\right)^n.$$

Then $k\,T_{ij/R} = (u_{ij+1} - u_{ij}) - r\varepsilon_i\sigma_{ij}$ (47)

and $k\,T_{i0/R} = (u_{i,1} - u_{i0}) - r\varepsilon_i\sigma_{i0}$ (48)

eliminating ε_i between (4.43) and (4.44) we get

$\sigma_{i0}\,T_{ij/R} = T_{ij/C}\,\sigma_{i0} - T_{i0/C}\,\sigma_{ij}$. (49)

Then

$$\sigma_{i0}T_{ij/R} = \sum_{n=0}^{\infty}\sum_{r=0}^{n} \alpha_r\beta_{n-r}\left[\left(u_{t^{r+1}}\right)_{ij}\left(u_{t^{n-r}}\right)_{i0} - \left(u_{t^{r+1}}\right)_{i0}\left(u_{t^{n-r}}\right)_{ij}\right]. \quad (50)$$

Putting $A_{nr} = \left[\left(u_{t^{r+1}}\right)_{ij}\left(u_{t^{n-r}}\right)_{i0} - \left(u_{t^{r+1}}\right)_{i0}\left(u_{t^{n-r}}\right)_{ij}\right]$,

then for hyperbolic problems of continuous exact solution $u(x,t) = e^{-c^2t} f(x)$, we can show that $A_{nr} = 0$.

Conclusion

Let $F_{ij}(u) = 0$ represents the difference equation approximating the PDE at the grid point *(ih, jk)*. If we replace the exact solution u of the difference equation by the exact solution *U* of the PDE, the value of $F_{ij}(U)$ is called the local truncation error at the grid point *(ih,jk)* and is denoted by T_{ij}.

In order to obtain the order of the local truncation error T_{ij} one have to use Taylor's expansion about the point *(ih, jk)* for each term in $F_{ij}(U)$ and use the original PDE. The principal part of the local truncation error will indicate its order. The finite difference method is accurate of order *(p, q)* if its local truncation error T_{ij} is of order *(p, q)* i.e. $T_{ij} = 0(h^p, k^q)$.

The Round-Off Error:

The finite difference equations can not be solved exactly, even if it is marched, because the numerical computation is carried out only up to a finite number of

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decimal places. Consequently, another kind of error is introduced in the finite difference solution during the actual process of computation. This error is called the round-off error, if the actual computation solution is□u_{ij} then

u_{ij} -□u_{ij} = r_{ij}

is the global round-off error at the grid point *(ih, jk)*

Consistency:

The difference equations is said to be consistent with the PDE, if the local truncation error T_{ij} tends to zero as the mesh lengths tend to zero

Stability:

If the exact solution of PDE is bounded, then the numerical solution must also be bounded. The concept of stability applies in this case is defined as follows:

"when applied to a PDE that has a bounded solution, a finite difference equation is stable if it produces a bounded solution and is unstable if it produces an unbounded solution."

If the exact solution of the PDE is unbounded then the numerical solution also must be unbounded. The concept of stability doesn't apply in that case.

Convergence:

A finite difference method is convergent if the solution of PDE approaches the exact solution of the PDE as the sizes of the grid spacing tend to zero.

Theorem (Lax equivalence theorem)

Given a well-posed linear-initial-value problem and a finite difference approximation to it that is consistent, stability is the necessary and sufficient condition for convergence.

There are many other methods for solving PDEs, such as Multigrid, Finite Elements, Boundary Elements, Finite Volume, Flux and Domain Decomposition methods.

References

[1][] Hassan N. A. Ismail "On The convergence of The Restrictive Pade` Approximation to The Exact Solutions of IBVP of Parabolic and Hyperbolic Types" Article in Press Appl. Math. and Computation

مجلة جامعة فزان العلمية
Fezzan University scientific Journal
Journal homepage: wwwhttps://fezzanu.edu.ly/

[2][] Hassan N. A. Ismail, Ghada S. E. Salem " On The convergence of The Restrictive Taylor Approximation to The Exact Solutions of IBVP of Parabolic and Hyperbolic, Convection Diffusion, and KdV Equations, 29th International Conference on statistics, Computer Science & Applications Vol. 2, pp.11-22 Cairo (2004). Sent for Publication for Appl. Math. and Computation, (U.S.A).

[3][] Elsayed M. E. Elbarbary “Accelerating Technique for Implicit Finite Difference Schemes for Two Dimensional Parabolic Partial Differential Equations” Int. J. Comp. Math. Vol. 77, pp 135-144 (2001).

[4][] B. Cockburn, C. Johnson, C.-W. Shu and E. Tadmor, Advanced Numerical Approximation of Nonlinear Hyperbolic Equations, C.I.M.E. Course in Cetraro, Italy, June 1997, ed. A. Quarteroni, Lecture Notes in Mathematics, Vol. 1697 (Springer, New York, 1998).

[5][] A. de Pablo, F. Quir´os, and A. Rodr´ıguez. Nonlocal filtration equations with rough kernels. Nonlinear Anal., 137:402–425, 2016.

[6][] N. Alibaud and B. Andreianov. Non-uniqueness of weak solutions for the fractal Burgers equation. Ann. Inst. H. Poincar´e Anal. Non Lin´eaire, 27(4):997– 1016, 2010.

[7][] Jain M.K. & Iyengar S.R.K. and Jain R. K. “Computational Methods for partial Differential Equations” Wiley Eastern Limited, (1994).

[8][] Randall J. Le Veque “Numerical Methods for Conservation laws” (Second Edition) Birkhauser Verlage Berlin, (1992).