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Fekete-Szegö Inequality for a Class of Bi-univalent Functions Involving Euler Polynomials and Hurwitz–Lerch Zeta Function

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Abstract
In this study, we introduce a new subclass of bi-univalent functions denoted by $\mathcal{F}_{\sum}^{\alpha,b.s}(\lambda,\delta,v)$, using Euler polynomials and Hurwitz-Lerch Zeta function. Subsequently, an estimation of coefficient bounds and the Fekete-Szegö inequality. Additionally, some new results are shown after specializing in the parameters employed in the main results.

Keywords: Analytic function, Univalent and bi-univalent functions, Euler polynomials, Fekete-Szegö, Hurwitz-Lerch Zeta function.

1.Introduction and Previous Studies

The theory of univalent functions has been one of the central topics in geometric function theory, owing to its deep connections with complex analysis and its numerous applications. An analytic function f in the open unit disk $U=\{z\in C:|z|<1\}$ is said to be a univalent function if it is one-to-one; the class of all such functions is usually denoted by S (see [1]). In 1967, Lewin introduced the concept of bi-univalent functions as a natural extension of the popular class of univalent functions S. A function fis said to be bi-univalent if both f and its inverse are univalent in the open unit disk U, which is denoted by $\sum$. Recently, many authors have studied this type of class and investigated upper bounds for the coefficients and the Fekete–Szegö inequality of functions belonging to various subclasses of bi-univalent functions (see [2–6]).

Motivated by the important role played by special polynomials in defining various subclasses of univalent functions and bi-univalent functions, we turn our attention to Euler polynomials, these polynomials, which originate from the pioneering work of Leonhard Euler in the eighteenth century, In recent years, Euler polynomials have attracted considerable interest in geometric function theory, particularly in the study of analytic, univalent and bi-univalent functions. Due to their wide applicability, several authors have derived sharp coefficient estimates for different subclasses of analytic and bi- univalent functions associated with Euler polynomials. For instance, Amourah et al. [7] and Frasin et al. [8]. In this paper, the researcher's generalization a new subclass of the bi-univalent function by Hurwitz- Lerch Zeta function and Euler polynomials, estimated the coefficients bounds for |a_2 |and |a_3 | and solve Fekete-Szegö |a_3-μa_2^2 |.

Let $\mathcal{A}$ denote the class of all analytic functions in the open unit disk $\mathbb{U}$ and normalized by $f(0)=f'(0)-1=0$ of the form:

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k. \qquad (1)$$

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The Koebe one-quarter theorem see [1] offer that every univalent function $f \in S$ contains a disk of radius $\left(\frac{1}{4}\right)$, the inverse of $f \in \mathbb{U}$ is a univalent analytic function defined on the disk by $\mathbb{U}_p := \left\{z: z \in \mathbb{C} \text{ and } |z| < p; p \geq \frac{1}{4}\right\}$.

Therefore, for each function $f(z) = w \in S$, there is an inverse function $f^{-1}(w)$ of $f(z)$ defined by

$$f^{-1}(f(z)) = z\ (z \in U) \text{and } f(f^{-1}(w)) = w\ (w \in \mathbb{U}_p),$$

where

$$g(w) = f^{-1}(w)$$
$$= w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \quad (2)$$

An example of a function belonging to the class $\sum$ is $h(z) = \frac{z}{1-z}$ but $h(z) = \frac{z}{1-z^2}$ does not belong to the class $\sum$ (see [9]).

The Euler polynomials $\Phi_i(v)$, similar to other polynomial families, can be derived using specific generating function (see [11,12]) as the following:

$$K(v,h) = \frac{2e^{hv}}{e^h + 1}$$
$$= \sum_{i=0}^{\infty} \Phi_i(v) \frac{h^i}{i!}, \left(\frac{1}{2} < v \leq 1, |h| < \pi\right),$$

an explicit formula for $\Phi_i(v)$ is given by

$$\Phi_i(v) = \sum_{i=0}^{i} \frac{1}{2^i} \sum_{u=0}^{i} (-1)^u \binom{i}{u} (u+v)^j. \quad (3)$$

Now $\Phi_i(v)$ in terms of Φ_u , obtained from (3) as:

$$\Phi_i(v) = \sum_{u=0}^{i} \frac{\phi_u}{2^u} \binom{i}{u} \left(v - \frac{1}{2}\right)^{i-u}.$$

Initial Euler polynomial values are:

$$\begin{aligned} &\Phi_0(v) = 1; \\ &\Phi_1 = \frac{2v-1}{2}; \\ &\Phi_2 = v^2 - v; \\ &\Phi_3 = \frac{4v^3 - 6v^2 + 1}{4}; \\ &\Phi_4 = v^4 - 2v^3 + v. \end{aligned} \quad (4)$$

Definition 1.1.[10] For analytic functions f and g in the unit disk $\mathbb{U}$, we say that a function f is subordinate to a function g, dented by $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function $w \in \mathcal{A}$ in the unit disk $\mathbb{U}$ such that $f(z) = g(w(z)), (z \in \mathbb{U})$. Also, if g is univalent in $\mathbb{U}$, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

2.Materials and Methods

Lemma 2.1. [1] If $p \in P$,then $|c_n| \leq 2$ for each $n \in \mathbb{N}$, where P is the family of analytic functions in $\mathbb{U}$ such that

$$\operatorname{Re}\{p(z)\} > 0,\ p(z) = 1 + c_1 z + c_2 z^2 + \cdots,\ (z \in \mathbb{U}).$$

Nagat & Darus ([14,15]) introduced the generalized integral operator associated with the general Hurwitz- Lerch Zeta function, denoted by $\mathfrak{J}^{\alpha}_{s,b}f(z)$ for $f \in \mathcal{A}$ as follows:

For $s \in \mathbb{C}, b \in \mathbb{C} - \mathbb{Z}_0^-$ the generalized integral operator $\mathfrak{J}^{\alpha}_{s,b}f(z): \mathcal{A} \to \mathcal{A}$ is defined by

$$\mathfrak{J}^{\alpha}_{s,b}f(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)} \left(\frac{b}{k-1+b}\right)^s a_k z^k \ (z \in \mathbb{U}, \alpha \neq 2,3,4,\dots). \quad (5)$$

Many other works on analytic and univalent functions related to this operator can be see ([16]- [19]).

By using a generalized integral operator $\mathfrak{J}^{\alpha}_{s,b}f(z)$ in (5), a new subclass of bi-univalent functions is considered as the following.

For $s \in \mathbb{C}, b \in \mathbb{C} - \mathbb{Z}_0^-$ and $\alpha \neq 2,3,4,\dots$, a function $f \in \sum$ and of the form (1) is said to be in the subclass $\mathcal{F}_{\Sigma}^{\alpha,b.s}(\lambda,\delta,v)$if the following conditions are satisfied:

$$\mathrm{Re}\left\{\frac{(1-\lambda)\left(\mathfrak{J}^{\alpha}_{s,b}f(z)\right)}{z} + \lambda\left(\mathfrak{J}^{\alpha}_{s,b}f(z)\right)` + \delta z(\mathfrak{J}^{\alpha}_{s,b}f(z))"\right\} \prec K(v,z) =$$
$$\sum_{i=0}^{\infty} \Phi_i(v)\ \frac{z^i}{i!}, \qquad (6)$$

and

$$\mathrm{Re}\left\{\frac{(1-\lambda)\left(\mathfrak{J}^{\alpha}_{s,b}g(w)\right)}{w} + \lambda\left(\mathfrak{J}^{\alpha}_{s,b}g(w)\right)` + \delta w(\mathfrak{J}^{\alpha}_{s,b}g(w))"\right\} \prec K(v,w) =$$
$$\sum_{i=0}^{\infty} \Phi_i(v)\ \frac{w^i}{i!}, \quad (7)$$

where $\lambda \geq 1\ \delta \geq 0,\ \frac{1}{2} < v \leq 1\ z,w\ \in \mathbb{U}$ and $g = f^{-1}$.

It is of interest to note that by taking $\alpha = 0$ and $s = 0$ in the subclass $\mathcal{F}_{\Sigma}^{\alpha,b.s}(\lambda,\delta,v)$, we state the following subclass $\mathcal{F}_{\Sigma}(\lambda,\delta,v)$.

Remark 2.2.[7] If the next subordinations are satisfied for a function $f \in \mathbb{U}$ given

$$\mathrm{Re}\left\{(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) + \delta z f"(z)\right\} \prec K(v,z) = \sum_{i=0}^{\infty} \Phi_i(v)\ \frac{z^i}{i!},$$

and

$$\mathrm{Re}\left\{(1-\lambda)\frac{g(w)}{w} + \lambda g'(w) + \delta w g"(w)\right\} \prec K(v,w) = \sum_{i=0}^{\infty} \Phi_i(v)\ \frac{w^i}{i!},$$

where $\lambda \geq 1\ \delta \geq 0,\ \frac{1}{2} < v \leq 1\ z,w\ \in \mathbb{U}$ and $g = f^{-1}$.

In mathematics, the Fekete–Szegő is an inequality for the coefficients of univalent functions found by Fekete and Szegő [13] for the estimates of $|a_3 - \mu a_2^2|$ when $a_1 = 1$ with μ real. The well-known result due to them states that if $f \in \mathcal{A}$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 4\mu - 3 & \text{if } \mu \geq 1, \\ 1 + 2\exp\left(\frac{-2\mu}{1-\mu}\right) & \text{if } 0 \leq \mu \leq 1, \\ 3 - 4\mu & \text{if } \mu \leq 0. \end{cases}$$

3.Results and Discussion

For a function $f \in \sum$, we provide the coefficient estimations and solve Fekete-Segö for a new class $\mathcal{F}_{\Sigma}^{\alpha,b.s}(\lambda,\delta,v)$ in the following theorem.

Theorem 3.1. Let $f \in \sum$ given by (1) and belong to the class $\mathcal{F}_{\Sigma}^{\alpha,b.s}(\lambda,\delta,v)$, where $\lambda \geq 1$,$\delta \geq 0$, $\frac{1}{2} < v \leq 1$, $z,w \in \mathbb{U}$ and $g = f^{-1}$ .Then

$$|a_2| \leq \sqrt{\frac{(2v-1)^3}{2\left|(1+2\lambda+6\delta)(2v-1)^2\varphi_3^{\alpha,b,s}-2(1+\lambda+2\delta)^2(v^2-3v+1)\left(\varphi_2^{\alpha,b,s}\right)^2\right|}},$$

$$|a_3| \leq \frac{2v-1}{2(1+2\lambda+6\delta)\varphi_3^{\alpha,b,s}} + \frac{(2v-1)^2}{4(1+\lambda+2\delta)^2\left(\varphi_2^{\alpha,b,s}\right)^2},$$

and

$$|a_3-\mu a_2^2| \leq$$

$$1\begin{cases} \dfrac{2v-1}{(1+2\lambda+6\delta)\varphi_3^{\alpha,b,s}}, \text{if } \ 0 \leq |1-\mu| \dfrac{(2v-1)^3}{2\left|(1+2\lambda+6\delta)(2v-1)^2\varphi_3^{\alpha,b,s}-2(1+\lambda+2\delta)^2(v^2-3v+1)\left(\varphi_2^{\alpha,b}\right.\right.\cdots} \\ 2|1-\mu| \dfrac{(2v-1)^3}{2\left|(1+2\lambda+6\delta)(2v-1)^2\varphi_3^{\alpha,b,s}-2(1+\lambda+2\delta)^2(v^2-3v+1)\left(\varphi_2^{\alpha,b,s}\right)^2\right|}, \text{ if } \ |1-\mu| \dfrac{(2v-\cdots}{2\left|(1+2\lambda+6\delta)(2v-1)^2\varphi_3^{\alpha,b,s}-2(1\cdots\right.} \end{cases}$$

where

$\varphi_{k=2}^{\alpha,b,s} = \frac{\Gamma(3)\Gamma(2-\alpha)}{\Gamma(3-\alpha)}\left(\frac{b}{b+1}\right)^s$, and $\varphi_{k=3}^{\alpha,b,s} = \frac{\Gamma(4)\Gamma(2-\alpha)}{\Gamma(4-\alpha)}\left(\frac{b}{b+2}\right).^s$

Proof.

Let $f(z) \in \mathcal{F}_{\Sigma}^{\alpha,b.s}(\lambda,\delta,v)$, from (6) and (7), we have

$$(1-\lambda)\frac{\mathfrak{J}_{s,b}^{\alpha}f(z)}{z} + \lambda(\mathfrak{J}_{s,b}^{\alpha}f(z))' + \delta z(\mathfrak{J}_{s,b}^{\alpha}f(z))'' \prec K(v,z).$$

by applying some calculating, we obtain

$$(1-\lambda)\frac{\mathfrak{J}_{s,b}^{\alpha}f(z)}{z} + \lambda(\mathfrak{J}_{s,b}^{\alpha}f(z))' + \delta z(\mathfrak{J}_{s,b}^{\alpha}f(z))''$$
$$= 1 + \frac{(1+\lambda+2\delta)\Gamma(3)\Gamma(2-\alpha)}{\Gamma(3-\alpha)}\left(\frac{b}{b+1}\right)^s a_2 z + \frac{(1+2\lambda+6\delta)\Gamma(4)\Gamma(2-\alpha)}{\Gamma(4-\alpha)}\left(\frac{b}{b+2}\right)^s a_3 z^2 + \cdots \prec K(v,s(z)) \qquad (8)$$

and

$$(1-\lambda)\frac{\mathfrak{J}_{s,b}^{\alpha}g(w)}{z} + \lambda(\mathfrak{J}_{s,b}^{\alpha}g(w))' + \delta w(\mathfrak{J}_{s,b}^{\alpha}g(w))'' \prec K(v,s(w))$$

Following the same procedure

$$(1-\lambda)\frac{\mathfrak{J}_{s,b}^{\alpha}g(w)}{z} + \lambda(\mathfrak{J}_{s,b}^{\alpha}g(w))' + \delta w(\mathfrak{J}_{s,b}^{\alpha}g(w))'' = 1 - \frac{(1+\lambda+2\delta)\Gamma(3)\Gamma(2-\alpha)}{\Gamma(3-\alpha)}\left(\frac{b}{b+1}\right)^s a_2 w + \frac{(1+2\lambda+6\delta)\Gamma(4)\Gamma(2-\alpha)}{\Gamma(4-\alpha)}\left(\frac{b}{b+2}\right)^s (2a_2^2-a_3)w^2 + \cdots \prec K(v,s(w)) \qquad (9)$$

Assume that there are two functions $r_1, r_2 : \mathbb{U} \to \mathbb{U}$ with $r_1(0) = r_2(0) = 0$ and $|r_1(z)| < 1, |r_2(w)| < 1$ for all $z,w \in \mathbb{U}$. So, we can define $\gamma, \sigma \in P$ as:

$$\gamma(z) = \frac{r_1(z)+1}{1-r_1(z)} = 1 + \gamma_1 z + \gamma_2 z^2 + \gamma_3 z^3 + \cdots, |\gamma_i| \leq 2, i \in \mathbb{N}.$$

then

$$r_1(z)=\frac{\gamma(z)-1}{\gamma(z)+1}$$
$$=\frac{\gamma_1}{2}z+\left(\frac{\gamma_2}{2}-\frac{\gamma_1^2}{4}\right)z^2+\frac{1}{2}\left(\gamma_3-\gamma_1\gamma_2+\frac{\gamma_1^3}{4}\right)z^3+\cdots \qquad (10)$$

and

$$\sigma(w)=\frac{r_2(w)+1}{1-r_2(w)}=1+\sigma_1 w+\sigma_2 w^2+\sigma_3 w^3+\cdots,|\sigma_i|\le 2, i\in\mathbb{N}.$$

then

$$r_2(w)=\frac{\sigma(w)-1}{\sigma(w)+1}=\frac{\sigma_1}{2}w+\left(\frac{\sigma_2}{2}-\frac{\sigma_1^2}{4}\right)w^2+\frac{1}{2}\left(\sigma_3-\sigma_1\sigma_2+\frac{\sigma_1^3}{4}\right)w^3+\cdots. \qquad (11)$$

Using (10) and (11), we get

$$K(v,r_1(z))=\Phi_0(v)+\frac{\Phi_1(v)}{2}\gamma_1 z+\left(\frac{\Phi_1(v)}{2}\left(\gamma_2-\frac{\gamma_1}{2}\right)+\frac{\Phi_2(v)}{8}\gamma_1^2\right)z^2+\left(\frac{\Phi_1(v)}{2}\left(\gamma_3-\gamma_1\gamma_2+\frac{\gamma_1^3}{4}\right)+\frac{\Phi_2}{4}\left(\gamma_1\gamma_2-\frac{\gamma_1^3}{2}\right)+\frac{\Phi_3(v)}{48}\gamma_1^3\right)z^3+\cdots \qquad (12)$$

and

$$K(v,r_2(w))=\sum_{i=0}^{\infty}\Phi_i(v)\frac{w^i}{i!}=\Phi_0(v)+\frac{\Phi_1(v)}{2}\sigma_1 w+\left(\frac{\Phi_1(v)}{2}\left(\sigma_2-\frac{\sigma_1}{2}\right)+\frac{\Phi_2(v)}{8}\sigma_1^2\right)w^2\left(\frac{\Phi_1(v)}{2}\left(\sigma_3-\sigma_1\sigma_2+\frac{\sigma_1^3}{4}\right)+\frac{\Phi_2}{4}\left(\sigma_1\sigma_2-\frac{\sigma_1^3}{2}\right)+\frac{\Phi_3(v)}{48}\sigma_1^3\right)w^3+\cdots \qquad (13)$$

From (6), (7) and the previous two equations (10), (11), we have

$$\frac{(1+\lambda+2\delta)\Gamma(3)\Gamma(2-\alpha)}{\Gamma(3-\alpha)}\left(\frac{b}{b+1}\right)^s a_2=\frac{\Phi_1(v)}{2}\gamma_1 \qquad (14)$$

$$\frac{(1+2\lambda+6\delta)\Gamma(4)\Gamma(2-\alpha)}{\Gamma(4-\alpha)}\left(\frac{b}{b+2}\right)^s a_3=\frac{\Phi_1(v)}{2}\left(\gamma_2-\frac{\gamma_1}{2}\right)+\frac{\Phi_2(v)}{8}\gamma_1^2 \qquad (15)$$

$$-\frac{(1+\lambda+2\delta)\Gamma(3)\Gamma(2-\alpha)}{\Gamma(3-\alpha)}\left(\frac{b}{b+1}\right)^s a_2=\frac{\Phi_1(v)}{2}\sigma_1 \qquad (16)$$

and

$$\frac{(1+2\lambda+6\delta)\Gamma(4)\Gamma(2-\alpha)}{\Gamma(4-\alpha)}\left(\frac{b}{b+2}\right)^s (2a_2^2-a_3)=\frac{\Phi_1(v)}{2}\left(\sigma_2-\frac{\sigma_1}{2}\right)+\frac{\Phi_2(v)}{8}\sigma_1^2 \qquad (17)$$

From (16), we have

$$(1+\lambda+2\delta)\,\varphi_2^{\alpha,b,s}a_2=-\frac{\Phi_1(v)}{2}\sigma_1$$

Adding two equations (14) and (16) and some simplifying, we obtain

$$\gamma_1=-\sigma_1 \quad \text{and} \quad \gamma_1^2=\sigma_1^2 \qquad (18)$$

and

$$2(1+\lambda+2\delta)^2\left(\varphi_2^{\alpha,b,s}\right)^2 a_2^2 = \frac{\Phi_1^2(v)}{4}(\gamma_1^2 + \sigma_1^2) \qquad (19)$$

$$a_2^2 = \frac{\Phi_1^2(v)(\gamma_1^2+\sigma_1^2)}{8(1+\lambda+2\delta)^2\left(\varphi_2^{\alpha,b,s}\right)^2}$$

Adding (15) to (17), we get

$$8(1+2\lambda+6\delta)\varphi_3^{\alpha,b,s}a_2^2 = 2\Phi_1(v)(\sigma_2+\gamma_2) + (\gamma_1^2+\sigma_1^2)\left(\frac{\Phi_2(v)}{2} - \Phi_1(v)\right)$$

by (18), we have

$$8(1+2\lambda+6\delta)\varphi_3^{\alpha,b,s}a_2^2 = 2\Phi_1(v)(\sigma_2+\gamma_2) + \gamma_1^2\big(\Phi_2(v) - 2\Phi_1(v)\big) \qquad (20)$$

Also, applying (18) in (19), we have

$$\gamma_1^2 = \frac{4(1+\lambda+2\delta)^2\left(\varphi_2^{\alpha,b,s}\right)^2 a_2^2}{\Phi_1^2(v)} \qquad (21)$$

Replacing γ_1^2 in (20)

$$a_2^2 = \frac{\Phi_1^3(v)(\sigma_2+\gamma_2)}{2\left[2(1+2\lambda+6\delta)\varphi_3^{\alpha,b,s}\Phi_1^2(v) - (1+\lambda+2\delta)^2\left(\varphi_2^{\alpha,b,s}\right)^2(\Phi_2(v)-2\Phi_1(v))\right]}$$

$$|a_2|^2 = \frac{\Phi_1^3(v)(|\sigma_2|+|\gamma_2|)}{2\left|2(1+2\lambda+6\delta)\varphi_3^{\alpha,b,s}\Phi_1^2(v) - (1+\lambda+2\delta)^2\left(\varphi_2^{\alpha,b,s}\right)^2(\Phi_2(v)-2\Phi_1(v))\right|}$$

Applying (4) and Lemma 2.1, we obtain

$$|a_2| \le \sqrt{\frac{(2v-1)^3}{2\left|(1+2\lambda+6\delta)\varphi_3^{\alpha,b,s}(2v-1)^2 - 2(1+\lambda+2\delta)^2\left(\varphi_2^{\alpha,b,s}\right)^2(v^2-3v+1)\right|}},$$

subtracting (17) from (15), then view (18) and after doing some calculations, we have

$$8(1+2\lambda+6\delta)\varphi_3^{\alpha,b,s}a_2^2 - 8(1+2\lambda+6\delta)\varphi_3^{\alpha,b,s}a_3 = 2\Phi_1(v)(\sigma_2-\gamma_2) + (\gamma_1^2-\sigma_1^2)\left(\frac{\Phi_2(v)}{2} - \Phi_1(v)\right)$$

then,

$$a_3 = a_2^2 + \frac{\Phi_1(v)(\sigma_2-\gamma_2)}{4(1+2\lambda+6\delta)\varphi_3^{\alpha,b,s}} \qquad (22)$$

From (19), we get

$$a_3 = \frac{\Phi_1^2(v)(\gamma_1^2)}{4(1+\lambda+2\delta)^2\left(\varphi_2^{\alpha,b,s}\right)^2} + \frac{\Phi_1(v)(\sigma_2-\gamma_2)}{4(1+2\lambda+6\delta)\varphi_3^{\alpha,b,s}}$$

Appling (4) and lemma 2.1, we have

$$|a_3| \le \frac{(2v-1)^2}{4(1+\lambda+2\delta)^2\left(\varphi_2^{\alpha,b,s}\right)^2} + \frac{2v-1}{2(1+2\lambda+6\delta)\varphi_3^{\alpha,b,s}},$$

From (22), we obtained

$$a_3-\mu a_2^2=\frac{\Phi_1(v)(\sigma_2-\gamma_2)}{4(1+2\lambda+6\delta)\varphi_3^{\alpha,b,s}}+(1-\mu)a_2^2$$

$$a_3-\mu a_2^2=\frac{\Phi_1(v)(\sigma_2-\gamma_2)}{4(1+2\lambda+6\delta)\varphi_3^{\alpha,b,s}}+\frac{(1-\mu)\Phi_1^3(v)(\sigma_2+\gamma_2)}{2\left[2(1+2\lambda+6\delta)\varphi_3^{\alpha,b,s}\Phi_1^2(v)-(1+\lambda+2\delta)^2\left(\varphi_2^{\alpha,b,s}\right)^2(\Phi_2(v)-2\Phi_1(v))\right]}$$

By using assist (4) in conjunction with the triangular inequality, we get:

$$|a_3-\mu a_2^2|\le\frac{2v-1}{2(1+2\lambda+6\delta)\varphi_3^{\alpha,b,s}}+\frac{|(1-\mu)|(2v-1)^3}{2\left|(1+2\lambda+6\delta)\varphi_3^{\alpha,b,s}(2v-1)^2-2(1+\lambda+2\delta)^2\left(\varphi_2^{\alpha,b,s}\right)^2(v^2-3v+1)\right|}$$

$$|a_3-\mu a_2^2|\le\frac{2v-1}{2(1+2\lambda+6\delta)\varphi_3^{\alpha,b,s}}+|(1-\mu)|\frac{(2v-1)^3}{2\left|(1+2\lambda+6\delta)(2v-1)^2\varphi_3^{\alpha,b,s}-2(1+\lambda+2\delta)^2(v^2-3v+1)\left(\varphi_2^{\alpha,b,s}\right)^2\right|}$$

If

$$|(1-\mu)|\frac{(2v-1)^3}{2\left|(1+2\lambda+6\delta)(2v-1)^2\varphi_3^{\alpha,b,s}-2(1+\lambda+2\delta)^2(v^2-3v+1)\left(\varphi_2^{\alpha,b,s}\right)^2\right|}\le\frac{2v-1}{2(1+2\lambda+6\delta)\varphi_3^{\alpha,b,s}},$$

we obtain

$$|a_3-\mu a_2^2|\le\frac{2v-1}{(1+2\lambda+6\delta)\varphi_3^{\alpha,b,s}}.$$

And if

$$|(1-\mu)|\frac{(2v-1)^3}{2\left|(1+2\lambda+6\delta)(2v-1)^2\varphi_3^{\alpha,b,s}-2(1+\lambda+2\delta)^2(v^2-3v+1)\left(\varphi_2^{\alpha,b,s}\right)^2\right|}\ge\frac{2v-1}{2(1+2\lambda+6\delta)\varphi_3^{\alpha,b,s}},$$

we obtain

$$|a_3-\mu a_2^2|\le 2|(1-\mu)|\frac{(2v-1)^3}{2\left|(1+2\lambda+6\delta)(2v-1)^2\varphi_3^{\alpha,b,s}-2(1+\lambda+2\delta)^2(v^2-3v+1)\left(\varphi_2^{\alpha,b,s}\right)^2\right|}.$$

This completes the proof.

When we set $\varphi_2^{\alpha,b,s}=1$, and $\varphi_3^{\alpha,b,s}=1$ in Theorem 3.1, we have

Corollary 3.2. [7] Let $f\in\Sigma$ given by (1) be in the class $\mathcal{F}_{\Sigma}^{0,b.0}(\lambda,\delta,v)$ where, $\lambda\ge0,\delta\ge0,\frac{1}{2}<v\le1$ $z,w\in\mathbb{U}$ and $g=f^{-1}$,then

$$|a_2|\le\sqrt{\frac{(2v-1)^3}{2|(1+2\lambda+6\delta)(2v-1)^2-2(1+\lambda+2\delta)^2(v^2-3v+1)|}},$$

$$|a_3|\le\frac{2v-1}{2(1+2\lambda+6\delta)}+\frac{(2v-1)^2}{4(1+\lambda+2\delta)^2},$$

and

$$|a_3-\mu a_2^2| \le \begin{cases} \frac{2v-1}{(1+2\lambda+6\delta)} & \text{if} \quad 0 \le |1-\mu| \frac{(2v-1)^3}{2|(1+2\lambda+6\delta)(2v-1)^2-2(1+\lambda+2\delta)^2(v^2-3v+1)|} \le \frac{2v-1}{2(1+2\lambda+6\delta)}, \\ 2|1-\mu| \frac{(2v-1)^3}{2|(1+2\lambda+6\delta)(2v-1)^2-2(1+\lambda+2\delta)^2(v^2-3v+1)|} & \text{if } |1-\mu| \frac{(2v-1)^3}{2|(1+2\lambda+6\delta)(2v-1)^2-2(1+\lambda+2\delta)^2(v^2-3v+1)|} \ge \frac{}{2} \end{cases}$$

When we set $\lambda = 1$ in corollary 3.2, we get the next remark.

Remark 3.3. [7] Let $f \in \sum$ given by (1) in the class $\mathcal{F}_{\Sigma}^{0,b.0}(1,\epsilon,v)$ where, $\delta \ge 0, \frac{1}{2} < v \le 1$ $z, w \in \mathbb{U}$ and $g = f^{-1}$,then

$$|a_2| \le \sqrt{\frac{(2v-1)^3}{2|3(1+2\delta)(2v-1)^2-8(1+\delta)^2(v^2-3v+1)|}},$$

$$|a_3| \le \frac{2v-1}{6(1+2\delta)} + \frac{(2v-1)^2}{16(1+\delta)^2} ,$$

and

$$|a_3-\mu a_2^2| \le \begin{cases} \frac{2v-1}{3(1+2\delta)} & \text{if} \quad 0 \le |1-\mu| \frac{(2v-1)^3}{2|3(1+2\delta)(2v-1)^2-8(1+\delta)^2(v^2-3v+1)|} \le \frac{2v-1}{6(1+2\delta)}, \\ 2|1-\mu| \frac{(2v-1)^3}{2|3(1+2\delta)(2v-1)^2-8(1+\delta)^2(v^2-3v+1)|} & \text{if} \quad |1-\mu| \frac{(2v-1)^3}{2|3(1+2\delta)(2v-1)^2-8(1+\delta)^2(v^2-3v+1)|} \ge \frac{2v-1}{6(1+2\delta)}. \end{cases}$$

When we set $\delta = 0$ in corollary 3.2, we get the next remark.

Remark 3.4. [7] Let $f \in \sum$ given by (1) in the class $\mathcal{F}_{\Sigma}^{0,b.0}(\lambda,0,v)$ where, $\lambda \ge 0$, $\frac{1}{2} < v \le 1$ $z, w \in \mathbb{U}$ and $g = f^{-1}$,then

$$|a_2| \le \sqrt{\frac{(2v-1)^3}{2|3(1+2\lambda)(2v-1)^2-2(1+\lambda)^2(v^2-3v+1)|}},$$

$$|a_3| \le \frac{2v-1}{2(1+2\lambda)} + \frac{(2v-1)^2}{4(\lambda+1)^2},$$

and

$$|a_3-\mu a_2^2| \le \begin{cases} \frac{2v-1}{(1+2\lambda)} & \text{if} \quad 0 \le |1-\mu| \frac{(2v-1)^3}{2|3(1+2\lambda)(2v-1)^2-2(1+\lambda)^2(v^2-3v+1)|} \le \frac{2v-1}{2(1+2\lambda)}, \\ 2|1-\mu| \frac{(2v-1)^3}{2|3(1+2\lambda)(2v-1)^2-2(1+\lambda)^2(v^2-3v+1)|} & \text{if } |1-\mu| \frac{(2v-1)^3}{2|3(1+2\lambda)(2v-1)^2-2(1+\lambda)^2(v^2-3v+1)|} \ge \frac{2v-1}{2(1+2\lambda)} \end{cases}$$

When we set $\delta = 0$, and $\lambda = 1$ in Corollary 3.2, we get the following remark.

Remark 3.5. [7] Let $f \in \sum$ given by (1) in the class $\mathcal{F}_{\Sigma}^{0,b.0}(1,0,v)$ where, $\delta \ge 0, \frac{1}{2} < v \le 1$ $z, w \in \mathbb{U}$ and $g = f^{-1}$,then

$$|a_2| \le \sqrt{\frac{(2v-1)^3}{2|4v^2+12v-5|}},$$

$$|a_3| \le \frac{2v-1}{6} + \frac{(2v-1)^2}{16},$$

and

$$|a_3-\mu a_2^2| \leq \begin{cases} \dfrac{2v-1}{3} & \text{if } 0 \leq |1-\mu| \dfrac{(2v-1)^3}{2|4v^2+12v-5|} \leq \dfrac{2v-1}{6}, \\ 2|1-\mu| \dfrac{(2v-1)^3}{2|4v^2+12v-5|} & \text{if } |1-\mu| \dfrac{(2v-1)^3}{2|4v^2+12v-5|} \geq \dfrac{2v-1}{6} \end{cases}$$

4.Conclusion

Special functions and polynomials are used in many differential mathematical and scientific fields that have recently been researched. Then, finding the upper bounds. Motivated by these developments, the present work introduces a new subclass of bi-univalent functions associated with Euler polynomials through a generalized integral operator. By employing this operator, we derive upper bound for $|a_2|, |a_3|$ and $|a_3-\mu a_2^2|$

Moreover, the results highlight the effectiveness of Euler polynomials in addressing coefficient-related problems for bi-univalent functions, leaving further sharpness investigations as an open problem for future research.

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