مجلة جامعة فزان العلمية
Fezzan University scientific Journal
journal@fezzanu.edu.ly

Meromorphic Uniformly β −Starlike p-Valent Functions with Positive Coefficients and Prescribed Second Coefficient.

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Abstract

In this work, we investigate a family of meromorphic p-valent functions defined in the punctured unit disk with prescribed positive coefficients. New coefficient bounds are derived for this class using analytic techniques adapted to uniformly β-starlike functions. In addition, we establish that the proposed class is invariant under arithmetic means and convex linear combinations. Finally, the radius of convexity is determined, and it is shown that the obtained results are sharp.

Keywords: Meromorphic, Regular, p-valent, Starlike, uniformly.

Introduction and Previous Studies

Let Σ_p denote the class of functions of the form

$$\psi(\xi)=\xi^{-p}+\sum_{n=1}^{\infty}a_{p+n-1}\xi^{p+n-1}(p\in N=\{1,2,\dots\}) \tag{1.1}$$

which are regular and p-valent in the punctured unit disc $U^*=\{\xi\ :\ 0<|\xi|<1\}$. Furthermore, we consider the subclass $\Sigma_p^*\subset\Sigma_p$ defined via functions that can be written in the form

$$\psi(\xi)=\xi^{-p}+\sum_{n=1}^{\infty}a_{p+n-1}\xi^{p+n-1}\left(a_{p+n-1}\geq 0\right). \tag{1.2}$$

A function $\psi(\xi)\in\Sigma_p$ is said to be in the class $\Sigma_p^*(p,\alpha,\beta)$ if

$$-Re\left\{\frac{\xi\psi'(\xi)}{\psi(\xi)}+\alpha\right\}>\beta\left|\frac{\xi\psi'(\xi)}{\psi(\xi)}+p\right|$$

for $0\leq\alpha<p$ and $\beta\geq 0$.

Meromorphic univalent functions have been extensively studied by Aouf and Silverman [1], Pommerenke [10], Clunie [4], Libera [6], Padmanabhan [9], Mogra et al. [8] and Uralegaddi [12]. Meromorphic p-valent functions have been extensively studied by Joshi and Aouf [5], Aouf et al. [2,3], Uralegaddi and Ganigi [13] and Mogra [7].

Lemma 1. Let $\psi(\xi)$ be defined as in (1.2). A necessary and sufficient condition for $\psi(\xi)\in\Sigma_p^*(p,\alpha,\beta)$ is that the coefficient inequality

$$\sum_{n=1}^{\infty}[(n+p)(1+\beta p)+\alpha-1]a_{p+n-1}\leq p-\alpha. \tag{1.3}$$

holds. The obtained condition is the best possible.

Proof. Let the condition (1.3) holds true and using the fact -Re($\omega)\geq\beta e^{i\theta}(\omega+p)+\alpha$ if and only if

$$-Re\left\{\left(1+\beta e^{i\theta}\right)\omega+p\beta e^{i\theta}\right\}\geq\alpha,$$

we have

*Corresponding author: medrbuk@elmergib.edu.ly Received in 9 February 2026
Revised form 8 march 2026 Accepted 10 march 2026 - Publication date 12 march 2026

$$-Re\left\{\frac{\xi\psi'(\xi)}{\psi(\xi)}+\alpha\right\}\geq\beta\left|\frac{\xi\psi'(\xi)}{\psi(\xi)}+p\right|,$$

Hence

$$-Re\left\{\left(1+\beta e^{i\theta}\right)\frac{\xi\psi'(z)}{\psi(\xi)}+p\beta e^{i\theta}\right\}\geq\alpha,$$

or, equivalently,

$$-Re\left\{\frac{\left(1+\beta e^{i\theta}\right)(\xi\psi'(\xi))+p\beta e^{i\theta}\psi(\xi)}{\psi(\xi)}\right\}\geq\alpha,$$

where $-\pi\leq\theta<\pi$. Suppose that
$\emptyset(\xi)=-\left(1+\beta e^{i\theta}\right)\xi\psi'(\xi)-p\beta e^{i\theta}\psi(\xi),$
$H(\xi)=\psi(\xi),$
and using the fact that
$-Re(\omega)\geq\alpha$ if and only if $|\omega-(p+\alpha)|\leq|\omega+(p-\alpha)|$ where
$\omega=-(u+iv),$ we need to prove that
$|\emptyset(\xi)+(p-\alpha)H(\xi)|\geq|\emptyset(\xi)-(p+\alpha)H(\xi)|$ for $0\leq\alpha<p.$
Then $|\emptyset(\xi)+(p-\alpha)H(\xi)|-|\emptyset(\xi)-(p+\alpha)H(\xi)|=$

$$\left|\frac{2p-\alpha}{z^p}-\sum_{n=1}^{\infty}[(n+p-1)-(p-\alpha)]a_{n+p-1}\xi^{n+p-1}-p\beta e^{i\theta}\sum_{n=1}^{\infty}(n+p)a_{n+p-1}\xi^{n+p-1}\right|$$

$$-\frac{-\alpha}{|\xi|^p}-\sum_{k=n}^{\infty}(1-\lambda+\lambda k)[k-(p-\alpha)]|a_k||\xi|^k-\beta\sum_{k=n}^{\infty}(k+p)(1-\lambda+\lambda k)|a_k||\xi|^k-$$

$$\frac{(1-\lambda p-\lambda)(-\alpha)}{|\xi|^p}-\sum_{k=n}^{\infty}(1-\lambda+\lambda k)[k+(p+\alpha)]|a_k||\xi|^k-\beta\sum_{k=n}^{\infty}(k+p)(1-\lambda+\lambda k)|a_k||\xi|^k$$

$$=\frac{2(p-\alpha)(1-\lambda p-\lambda)}{|\xi|^p}-2\sum_{k=n}^{\infty}(1-\lambda+\lambda k)(k+\alpha)|a_k||\xi|^k-2\beta\sum_{k=n}^{\infty}(k+p)(1-\lambda+\lambda k)|a_k||\xi|^k\geq 0$$

$$=\sum_{k=n}^{\infty}[1+\lambda(k-1)][k(1+\beta)+(\alpha+p\beta)]a_k\leq(p-\alpha)(1-\lambda p-\lambda)$$

Conversely, suppose that f is in the class $M(f;\alpha,\beta,\lambda).$ Then

$$-Re\left\{\left(1+\beta e^{i\theta}\right)\frac{[\xi\psi'(\xi)+\lambda\xi^2f''(\xi)]+p\beta e^{i\theta}[(1-\lambda)\psi(\xi)+\lambda\xi\psi'(\xi)]}{(1-\lambda)\psi(\xi)+\lambda\xi\psi'(\xi)}\right\}\geq\alpha,$$

Hence

$$Re\left\{\frac{(p-\alpha)(1-\lambda-\lambda p)\frac{1}{\xi^p}-\sum_{k=n}^{\infty}\{k+\beta e^{i\theta}(k+p)+\alpha\}[1+\lambda(k-1)]a_k\xi^k}{(1-\lambda-\lambda p)\frac{1}{\xi^p}+\sum_{k=n}^{\infty}[1+\lambda(k-1)]a_k\xi^k}\right\}\geq 0,$$

If we now choose $\xi\to 1^-$, we write

$\sum_{k=n}^{\infty}\{k(1+\beta)+(\alpha+p\beta)\}[1+\lambda(k-1)]a_k \leq (p-\alpha)(1-\lambda-\lambda p)$ ■

As a direct consequence of Lemma 1, the function $\psi(\xi)$ given by (1.2) is contained in the class $\Sigma_p^*(p,\alpha,\beta)$ satisfy

$$a_p \leq \frac{p-\alpha}{[(1+p)(1+\beta p)+\alpha-1]}. \qquad (1.4)$$

Here we may take

$$a_p = \frac{(p-\alpha)k}{[(1+p)(1+\beta p)+\alpha-1]}, 0 \leq k \leq 1. \qquad (1.5)$$

Let $\Sigma_{p,k}^*(p,\alpha,\beta,k)$ denote the subclass of $\Sigma_p^*(p,\alpha,\beta)$ consisting of the functions of the form

$$\psi(\xi) = \xi^{-p} + \frac{(p-\alpha)k}{[(1+p)(1+\beta p)+\alpha-1]}\xi^p + \sum_{n=2}^{\infty} a_{p+n-1}\xi^{p+n-1} \qquad (1.6)$$

where $a_{p+n-1} \geq 0$ and $0 \leq k \leq 1$.

Materials and Methods

The present study is concerned with deriving sharp coefficient inequalities for the class $\Sigma_{p,k}^*(p,\alpha,\beta,k)$. Furthermore, it is shown that this class is invariant under arithmetic means as well as convex linear combinations. In addition, the radius of convexity associated with this class is determined. All results obtained in this work are proved to be best possible. The analytical techniques employed are consistent with standard methods commonly used in the theory of univalent functions, see Silverman and Silvia [11].

Results and Discussion

2. Coefficient Inequalities.

Theorem 1. For the function $\psi(\xi)$ specified by equation (1.6), membership in the class $\Sigma_{p,k}^*(p,\alpha,\beta,k)$ is characterized by the condition below

$$\sum_{n=2}^{\infty}[(n+p)(1+\beta p)+\alpha-1]a_{p+n-1} \leq (p-\alpha)(1-k). \qquad (2.1)$$

Proof. Putting

$$a_p = \frac{(p-\alpha)k}{[(1+p)(1+\beta p)+\alpha-1]}, 0 \leq k \leq 1,$$

in (1.3) and simplifying we get the result.

Corolary 1. Let the function $\psi(\xi)$ defined by (1.6) be in the class $\Sigma_{p,k}^*(p,\alpha,\beta,k)$. Then

$$a_{p+n-1} \leq \frac{(p-\alpha)(1-k)}{[(n+p)(1+\beta p)+\alpha-1]} \qquad (n \geq 2). \qquad (2.2)$$

Sharpness of the result is demonstrated by the function

$$\psi(\xi) = \xi^{-p} + \frac{(p-\alpha)k}{[(1+p)(1+\beta p)+\alpha-1]}\xi^p + \frac{(p-\alpha)(1-k)}{[(n+p)(1+\beta p)+\alpha-1]}\xi^{p+n-1} (n \geq 2). \qquad (2.3)$$

3. Closure Theorems

This section is devoted to a rigorous analysis confirming that the class $\Sigma_{p,k}^*(p,\alpha,\beta,k)$ in question meets the proposed criteria is closed under arithmetic mean and convex linear combination.

Theorem 2. Let the functions

$$\psi_j(\xi) = \xi^{-p} + \frac{(p-\alpha)k}{[(1+p)(1+\beta p)+\alpha-1]}\xi^p + \sum_{n=2}^{\infty} a_{p+n-1}\xi^{p+n-1}((a_{p+n-1} \geq 0)\quad(3.1)$$

belong to the class $\Sigma_{p,k}^*(p,\alpha,\beta,k)$ for all $j = 1,2,\ldots,m$. Then the function

$$g(\xi) = \xi^{-p} + \frac{(p-\alpha)k}{[(1+p)(1+\beta p)+\alpha-1]}\xi^p + \sum_{n=2}^{\infty} b_{p+n-1}\xi^{p+n-1}((b_{p+n-1} \geq 0)$$
(3.2)

is also in the class $\Sigma_{p,k}^*(p,\alpha,\beta,k)$, where

$$b_{p+n-1} = \frac{1}{m}\sum_{j=1}^{m} a_{p+n-1,j}.$$
(3.3)

Proof. Since $\psi_j(\xi) \in \Sigma_{p,k}^*(p,\alpha,\beta,k)$ Applying Theorem 1, we obtain that

$$\sum_{n=2}^{\infty}[(n+p)(1+\beta p)+\alpha-1]a_{p+n-1,j} \leq (p-\alpha)(1-k)$$
(3.4)

for every $j = 1,2,\ldots,m$. Hence

$$\sum_{n=2}^{\infty}[(n+p)(1+\beta p)+\alpha-1]b_{p+n-1,j} = \sum_{n=2}^{\infty}[(n+p)(1+\beta p)+\alpha-1]\left[\frac{1}{m}\sum_{j=1}^{m} a_{p+n-1,j}\right]$$

$$= \frac{1}{m}\sum_{j=1}^{m}\sum_{n=2}^{\infty}[(n+p)(1+\beta p)+\alpha-1]a_{p+n-1,j} \leq (p-\alpha)(1-k)$$
(3.5)

and the result follows.

Theorem 3. Let

$$\psi_p(\xi) = \xi^{-p} + \frac{(p-\alpha)k}{[(1+p)(1+\beta p)+\alpha-1]}\xi^p$$
(3.6)

and

$$\psi_{p+n-1}(\xi) = \xi^{-p} + \frac{(p-\alpha)k}{[(1+p)(1+\beta p)+\alpha-1]}\xi^p + \frac{(p-\alpha)(1-k)}{[(n+p)(1+\beta p)+\alpha-1]}\xi^{p+n-1}(n \geq 2).$$
(3.7)

Then $\psi(\xi) \in \Sigma_{p,k}^*(p,\alpha,\beta,k)$ if and only if it is representable in the form

$$\psi(\xi) = \sum_{n=1}^{\infty} \lambda_{p+n-1}\psi_{p+n-1}(\xi),$$
(3.8)

where $\lambda_{p+n-1} \geq 0$ and $\sum_{n=1}^{\infty} \lambda_{p+n-1} = 1$

Proof. Let

$$\psi(\xi) = \sum_{n=1}^{\infty} \lambda_{p+n-1}\psi_{p+n-1}(\xi)$$

$$= \xi^{-p} + \frac{(p-\alpha)k}{[(1+p)(1+\beta p)+\alpha-1]}\xi^p + \sum_{n=2}^{\infty}\frac{(p-\alpha)(1-k)\lambda_{p+n-1}}{[(n+p)(1+\beta p)+\alpha-1]}\xi^{p+n-1}(n \geq 2)$$

Since

$$\sum_{n=2}^{\infty}\frac{(p-\alpha)(1-k)\lambda_{p+n-1}}{[(n+p)(1+\beta p)+\alpha-1]}\cdot\frac{[(n+p)(1+\beta p)+\alpha-1]}{(p-\alpha)(1-k)}$$
$$=\sum_{n=2}^{\infty}\lambda_{p+n-1}=1-\lambda_p\leq 1,$$

by Theorem 1, we have $\psi(\xi)\in\Sigma^*_{p,k}(p,\alpha,\beta,k)$.
Conversely, we suppose that $\psi(\xi)$ defined by (1.6) is in the class $\Sigma^*_{p,k}(p,\alpha,\beta,k)$.
Then by using (2.2), we get

$$a_{p+n-1}\leq\frac{(p-\alpha)(1-k)}{[(n+p)(1+\beta p)+\alpha-1]}\qquad(n\geq 2).$$

$$\lambda_{p+n-1}=\frac{[(n+p)(1+\beta p)+\alpha-1]}{(p-\alpha)(1-k)}a_{p+n-1}(n\geq 2)$$

and

$$\lambda_p=1-\sum_{n=2}^{\infty}\lambda_{p+n-1},$$

we have (3.8). This completes the proof of Theorem 3.
4. Radius of Convexity
Theorem 4. Let $\psi(\xi)$ be given by (1.6). If $\psi(\xi)\in\Sigma^*_{p,k}(p,\alpha,\beta,k)$, then $\psi(\xi)$ is meromorphically p-valent convex in $0<|\xi|<r=r(p,\alpha,\beta,k)$, where $r(p,\alpha,\beta,k)$ is the largest value for which

$$\frac{3p^2(p-\alpha)k}{[(1+p)(1+\beta p)+\alpha-1]}r^{2p}+\frac{(p+n-1)(3p+n-1)(1-k)}{[(1+p)(1+\beta p)+\alpha-1]}r^{2p+n-1}\leq p^2 \tag{4.1}$$

$(n=1,2,\dots)$.
The result is sharp for the function

$$\psi_{p+n-1}(\xi)=\xi^{-p}+\frac{(p-\alpha)k}{[(1+p)(1+\beta p)+\alpha-1]}\xi^p+\frac{(p-\alpha)(1-k)}{[(n+p)(1+\beta p)+\alpha-1]}\xi^{p+n-1}. \tag{4.2}$$

Proof. It suffices to establish that $\left|\frac{(\xi\psi'(\xi))'+p\psi'(\xi)}{\psi(\xi)}\right|\leq p$ for $0<|\xi|<r(p,\alpha,\beta,k)$.

Note that

$$\left|\frac{(\xi\psi'(\xi))'+p\psi'(\xi)}{\psi(\xi)}\right|$$

$$\leq\frac{\frac{2p^2(p-\alpha)k}{[(1+p)(1+\beta p)+\alpha-1]}r^{2p}+\sum_{n=2}^{\infty}(p+n-1)(2p+n-1)a_{p+n-1}r^{2p+n-1}}{p-\frac{p(p-\alpha)k}{[(1+p)(1+\beta p)+\alpha-1]}r^{2p}-\sum_{n=2}^{\infty}(p+n-1)a_{p+n-1}r^{2p+n-1}}\leq p \tag{4.3}$$

for $0<|\xi|<r\Leftrightarrow$

$$\frac{3p^2(p-\alpha)k}{[(1+p)(1+\beta p)+\alpha-1]}r^{2p}+\sum_{n=2}^{\infty}(p+n-1)(3p+n-1)a_{p+n-1}r^{2p+n-1}\leq p^2.$$

(4.4)

Since $\psi(\xi)$ is in the class $\Sigma_{p,k}^{*}(p,\alpha,\beta,k)$, from (2.1), we may take

$$a_{p+n-1}=\frac{(p-\alpha)(1-k)\lambda_{p+n-1}}{[(n+p)(1+\beta p)+\alpha-1]}\sum_{n=2}^{\infty}\lambda_{p+n-1}\leq 1.$$

(4.5)

Given a fixed r_0, let $n=n(r)$ denote the positive integer for which the expression $\frac{(3p+n-1)}{[(1+p)(1+\beta p)+\alpha-1]}r^{2p+n-1}$ is maximal. It follows that

$$\sum_{n=2}^{\infty}(p+n-1)(3p+n-1)a_{p+n-1}r^{2p+n-1}$$

$$\leq\frac{(3p+n-1)(p-\alpha)(1-k)}{[(1+p)(1+\beta p)+\alpha-1]}r^{2p+n-1}.$$

(4.6)

Next, determine the value $R_0=r_0\,(p,\alpha,\beta,k)$ along with the associated integer $n(r_0)$ so that

$$\frac{3p^2(p-\alpha)k}{[(1+p)(1+\beta p)+\alpha-1]}r_0^{2p}+\frac{(3p+n-1)(p-\alpha)(1-k)}{[(1+p)(1+\beta p)+\alpha-1]}r_0^{2p+n-1}=p^2.$$

It follows that r_0 serves as the radius of meromorphic p-valent convexity in the annulus
$0<|\xi|<r_0$.

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